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THE STABLE TOPOLOGY OF SELF-DUAL MODULI SPACES

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1. Introduction

Let M be a compact, oriented, Riemannian 4-manifold, and let G be a compact, simple Lie group. Consider a principal G-bundle, $P \to M$. Let $\mathscr{A}(P)$ denote the space of all smooth connections on P and let $\mathscr{G}(P)$ denote the space of smooth automorphisms of P. The group $\mathscr{G}(P)$ acts on $\mathscr{A}(P)$, but not always freely. However, $\mathscr{G}(P)$ does act freely on $P|_x \times \mathscr{A}(P)$, with $x \in M$ a fixed point; the quotient $\mathscr{B}'(P) \equiv (P|_x \times \mathscr{A}(P))/\mathscr{G}(P)$ defines a principal $\mathscr{G}(P)$-bundle. The space $\mathscr{B}'(P)$ has the weak homotopy type of the space $\mathrm{Maps}^*_P(M, BG)$ of smooth, based maps from M into BG (the classifying space for the group G) which pull-back the bundle P.

There exists a set of fiducial connections on P; these being the connections whose curvature 2-form is self-dual with respect to the Hodge star of the Riemannian metric (see, e.g., [2] or [13]). This set, $\mathfrak{m}(P)$, will be nonempty if the first Pontrjagin number of the associated bundle $\mathrm{Ad}\, P \equiv P \times_{\mathrm{Ad}\, G} \mathfrak{g}$ is sufficiently positive ($\mathfrak{g}$ denotes the Lie algebra of G). If said Pontrjagin number, $p_1(\mathrm{Ad}\, P)$, is negative, then $\mathfrak{m}(P)$ will be empty.

As $\mathfrak{m}(P)$ is invariant under the action of $\mathscr{G}(P)$, the quotient, $\mathfrak{M}'(P) \equiv (P|_x \times \mathfrak{m}(P))/\mathscr{G}(P)$ can be taken. Typically, the space $\mathfrak{M}'(P)$ sits in $\mathscr{B}'(P)$ as a real algebraic variety. When P and P' are isomorphic principal bundles, there are natural identifications of $\mathscr{B}'(P)$ and $\mathscr{B}'(P')$ which identify $\mathfrak{M}'(P)$ and $\mathfrak{M}'(P')$.

Simon Donaldson ([8], [9], [11], [12]) defined from $\mathfrak{M}'(P)$ an equivariant homology class in $\mathscr{B}'(P)$ (under that action of $G/\mathrm{Center}(G)$ which is induced by the action of G on $P|_x$). Suitably defined, this class is independent of the original choice of Riemannian metric on M and so depends only on the differential structure on M. Donaldson proved that there is a well-defined pairing of $\mathfrak{M}'(P)$ with suitable equivariant cocycles in $H^*(\mathscr{B}'(P))$. He (and now others) have exploited this discovery to build a profound and powerful tool with which to investigate the differential topology of 4-dimensional manifolds.

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Donaldson's program indicates that the inclusion map

$$i\colon \mathfrak{M}'(P) \to \mathscr{B}'(P) \tag{1.1}$$

induces a map of the respective homology and homotopy groups which is well worth studying.

When M is the 4-sphere with its standard metric, the effect of the map i on homology was first studied by Atiyah and Jones [3] who established that for $G = \mathrm{SU}(2)$, the induced map i_* on homology maps onto the homology of $\mathscr{B}'(P)$ up to a dimension which depends on the Pontrjagin number of $\operatorname{Ad} P$, and which increases as this Pontrjagin number gets large. Recently, Boyer and Mann [6] have obtained additional information on the homology of $\mathfrak{M}'(P)$ and the map i_* for $P \to S^4$ a principal SU(2)-bundle. Their results were obtained by defining and studying homology loop sum operations on a countable, disjoint union, $\bigcup_k \mathfrak{M}'(P_k)$, where $P_k \to S^4$ is a principal SU(2)-bundle with $4 \cdot k$ equaling the first Pontrjagin number of $\operatorname{Ad} P$. Frances Kirwan, using her technology for studying symplectic quotients [15], and Graham Segal [19], using techniques from analytic loop groups, have also studied the map i_* for $P \to S^4$ a principal G-bundle.

Still unproved is the eight-year-old conjecture of Atiyah and Jones; that the map i_* is an isomorphism on the respective q-dimensional homotopy groups for all q less than some $q(k)$ with $q(k)$ increasing with the instanton number k. However, Boyer and Mann have shown that $q(k) \leq k$.

The purpose of this article is to study the map i_* in the general context where M is not restricted to be S^4 with its standard metric, and where G is not restricted to be SU(2). The first result of this study is

Theorem 1. *Let M be a compact, connected, oriented, Riemannian 4-manifold, and let G be a simple, connected Lie group. Fix an integer $q < \infty$. Then, there exists $m(q) < \infty$ with the following significance: Let $P \to M$ be a principal G-bundle with $p_1(\operatorname{Ad} P) > m(q)$. Then the inclusion map i of (1.1) induces an epimorphism on the respective homology and homotopy groups in all dimensions less than or equal to q.*

The principal G-bundles over a compact, oriented 4-manifold are classified up to isomorphism by two characteristic classes. The first is the Pontrjagin number, $p_1(\operatorname{Ad} P)$. The second is a class, $\eta(P) \in H^2(M, \pi_1(G))$. When $p_1(\operatorname{Ad} P) = k$ and $\eta(P) = \eta$, it is convenient to denote $\mathscr{B}'(P)$ by $\mathscr{B}'(k, \eta)$ and $\mathfrak{M}'(P)$ by $\mathfrak{M}'(k, \eta)$ in order to stress the fact that these spaces depend only on the isomorphism class of P.

In §4, a positive constant $c(G)$ ($c(\mathrm{SU}(2)) = 4$) and natural homotopy equivalences between $\mathscr{B}'(k, \eta)$ and $\mathscr{B}'(k + c(G), \eta)$ are described. (These maps are the gluing maps which were introduced in [21] and exploited in [20]; see also §4 of [22].)

Theorem 2. *Let G be a compact, simple Lie group, and let M be a compact, connected, oriented Riemannian 4-manifold. Let (k,η) be characteristic classes for a principal G-bundle over M for which $\mathfrak{M}'(k,\eta)$ is not empty. There exists $j(k) \geq 0$ such that for any $j \geq j(k)$, $\mathfrak{M}'(k+c(G)\cdot j,\eta)$ is nonempty, and there exists a map of pairs*

$$T(j,k)\colon (\mathscr{B}'(k,\eta),\mathfrak{M}'(k,\eta)) \to (\mathscr{B}'(k+c(G)\cdot j,\eta),\ \mathfrak{M}'(k+c(G)\cdot j,\eta))$$

with the following properties:

(1) *$T(j,k)\colon \mathscr{B}'(k,\eta) \to \mathscr{B}'(k+c(G)\cdot j,\eta)$ is a homotopy equivalence.*

(2) *If $j_1 \geq j(k)$ and if $j_2 \geq j(k+c(G)\cdot j_1)$, then $T(j_2+j_1,k)$ is homotopic to $T(j_2,k+c(G)\cdot j_1)\circ T(j_1,k)$ as maps of pairs*

$$(\mathscr{B}'(k,\eta),\mathfrak{M}'(k,\eta)) \to (\mathscr{B}'(k+c(G)\cdot(j_1+j_2),\eta),\mathfrak{M}'(k+c(G)\cdot(j_1+j_2),\eta).$$

(3) *Let $z \in \pi_*(\mathscr{B}'(k,\eta),\mathfrak{M}'(k,\eta))$ or $z \in H_*(\mathscr{B}'(k,\eta),\mathfrak{M}'(k,\eta))$. There exists $J(z) \geq j(k)$ such that for all $j \geq J(z)$, $T(k,j)_*(z) = 0$ in*

$$\pi_*(\mathscr{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta))$$

or

$$H_*(\mathscr{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta)).$$

When the metric on TM is assumed to be generic in a suitable sense (see §7), Theorem 2 can be strengthened; the reader is referred to §7. The proofs of Theorems 1 and 2 are outlined in the next section, and occupy the remainder of this article.

Theorem 2 can be restated as a topological "stability" assertion. For this purpose, use the set of maps $\{T(k,j)\}$ of Theorem 2 to define the obvious direct limits of pairs:

$$(\mathscr{B}'(\infty,\eta),\mathfrak{M}'(\infty,\eta)) \equiv \operatorname*{dir\,lim}_{k\to\infty}(\mathscr{B}'(k,\eta)\mathfrak{M}'(k,\eta)). \tag{1.2}$$

Since each $T(k,j)$ is a homotopy equivalence, the space $\mathscr{B}'(\infty,\eta)$ has the weak homotopy type of any $\mathscr{B}'(k,\eta)$ with finite k.

Theorem 2*. *Let G be a compact, simple Lie group, and let M be a compact, oriented Riemannian 4-manifold. Let (k,η) be characteristic classes for a principal G-bundle over M. Using the set of maps $\{T(k,j)\}$ of Theorem 2, define the direct limit of pairs, $(\mathscr{B}'(\infty,\eta),\mathfrak{M}'(\infty,\eta))$. Then, the inclusion $\mathfrak{M}'(\infty,\eta) \subset \mathscr{B}'(\infty,\eta)$ induces a weak homotopy equivalence.*

A final remark: Assume that M is a complex manifold of real dimension 4 with a Kaehler metric. According to Donaldson [10], the moduli spaces of anti-self-dual connections on principal $\mathrm{U}(n)$ bundles over M, and the moduli spaces of holomorphic, and stable, rank-n vector bundles over M are essentially identical. For Kaehler M, Theorems 1 and 2 describe, via [10], the topology of the spaces of stable, holomorphic vector bundles over M with first Chern class zero. For stable, holomorphic bundles with nonzero first Chern class, construct from the rank-n vector bundle the principal $\mathrm{U}(n)$ bundle of frames, and then apply Theorems 1 and 2 to the moduli space of self-dual connections on the associated principal $\mathrm{PU}(n)$ bundle.

2. Strategy

Both Theorems 1 and 2 are proved via Morse theoretic arguments using the Yang-Mills functional. Fix a principal G-bundle $P \to M$, and let (k, η) denote the characteristic classes that classify P. By subtracting a multiple of $p_1(\mathrm{Ad}\, P)$ from the Yang-Mills functional, a function on $\mathscr{A}(P)$ is obtained which assigns to a connection A on P the number

$$\mathfrak{a}(A) \equiv \int_M |P_- F_A|^2, \tag{2.1}$$

where F_A denotes the curvature of the connection A, a section of the bundle $\Omega^2(\mathrm{Ad}\, P) \equiv \mathrm{Ad}\, P \otimes \bigwedge^2 T^*M$. (In general, $\Omega^k(\mathrm{Ad}\, P)$ denotes $\mathrm{Ad}\, P \otimes \bigwedge^k T^*M$.) The projection, $P_- \equiv (1 - *)/2$ is defined with the Riemannian metric's Hodge $*$-operator. Thus, $P_- F_A$ is a section of $P_-\Omega^2(\mathrm{Ad}\, P)$. (Also, $P_+ \equiv (1 + *)/2$).) The norm in (2.1) is defined via the Riemannian metric and the usual inner product on $\mathrm{Ad}\, P$ which is induced from the Killing form on $\mathfrak{g}$. The integration measure in (2.1) is defined by the Riemannian metric. The functional $\mathfrak{a}$ is $\mathscr{G}(P)$ equivariant, and so it descends to a functional on $\mathscr{B}'(k, \eta)$. By construction, $\mathfrak{a}^{-1}(0) = \mathfrak{M}'(k, \eta)$, and 0 is achieved by $\mathfrak{a}$ only when $\mathfrak{M}'(k, \eta)$ is nonempty.

Min-max theory for $\mathfrak{a}$ is described in §2 and in the Appendix of [22]. The theory concerns homotopy invariant families of compact subsets of $\mathscr{B}'(k, \eta)$, and more generally, for $\epsilon > 0$, families of compact subsets of $\mathscr{B}'(k, \eta)$ which are invariant under homotopies of $\mathscr{B}'(k, \eta)$ which map $\mathscr{B}'_\epsilon(k, \eta) \equiv \{b \in \mathscr{B}'(k, \eta) \colon \mathfrak{a}(b) < \epsilon\}$ into itself. For example, let $z \in H_m(\mathscr{B}'(k, \eta), \mathscr{B}'_\epsilon(k, \eta))$ be a relative class. This class can be represented by a singular m-dimensional chain in $\mathscr{B}'(k, \eta)$ whose boundary lies in $\mathscr{B}'_\epsilon(k, \eta)$. The class z defines a family $F(z)$ of compact subsets of $\mathscr{B}'(k, \eta)$ which are invariant rel $\mathscr{B}'_\epsilon(k, \eta)$: $F(z)$ is the family of compact singular chains which represent z. For a second example, let $z \in \pi_n(\mathscr{B}'(k, \eta), \mathscr{B}'_\epsilon(k, \eta))$; thus, z is represented by a map of the unit m-dimensional ball, B^n, into $\mathscr{B}'(k, \eta)$ which sends the boundary sphere

S^{n-1} into $\mathscr{B}'_\epsilon(k,\eta)$. Then,

$$F(z) \equiv \{\mathrm{Image}(\varphi)\colon \varphi \in C^0((B^n, S^{n-1}); (\mathscr{B}'(k,\eta), \mathscr{B}'_\epsilon(k,\eta))) \quad \text{and } [\varphi] = z\}.$$

For a given homotopy invariant (rel $\mathscr{B}'_\epsilon(k,\eta)$) family $\mathfrak{F}$, the min-max procedure attempts to produce a critical point of the Yang-Mills functional. Such a critical point may or may not exist; in either case, there is still the "critical value", the number

$$\mathfrak{a}_{\mathfrak{F}} \equiv \inf_{U \in \mathfrak{F}} \left\{ \sup_{b \in U} \mathfrak{a}(b) \right\}. \tag{2.2}$$

If $\mathfrak{a}_{\mathfrak{F}} > \epsilon$, then no set in $\mathfrak{F}$ lies in $\mathscr{B}'_\epsilon(k,\eta)$. If $\mathfrak{a}_{\mathfrak{F}} = \epsilon$, it may or may not be the case that a set in $\mathfrak{F}$ lies in $\mathscr{B}'_\epsilon(k,\eta)$ (see [22]).

Let $j > 0$ be an integer, and suppose that $T\colon \mathscr{B}'(k,\eta) \to \mathscr{B}'(k + c(G)\cdot j, \eta)$ is a homotopy equivalence. Let $\epsilon, \epsilon' > 0$ be given, and suppose that T maps $\mathscr{B}'_\epsilon(k,\eta)$ into $\mathscr{B}'_{\epsilon'}(k + c(G)\cdot j, \eta)$. If $\mathfrak{F}$ is a homotopy invariant (rel $\mathscr{B}'_\epsilon(k,\eta)$) family of compact subsets of $\mathscr{B}'(k,\eta)$, then T defines, by push forward, a homotopy invariant (rel $\mathscr{B}'_{\epsilon'}(k,\eta)$) family of compact subsets $T\mathfrak{F}$ of $\mathscr{B}'(k + c(G)\cdot j, \eta)$. It makes sense here to compare $\mathfrak{a}_{\mathfrak{F}}$ with $\mathfrak{a}_{T\mathfrak{F}}$.

The gluing construction of [20] as described in §4 of [22] provides a family of homotopy equivalences, $T\colon \mathscr{B}'(k,\eta) \to \mathscr{B}'(k + c(G), \eta)$. To describe these homotopy equivalences (see §4 for details), fix a base point $x_0 \in M$, and let $\mathfrak{P}(x_0)$ denote the space of maps from $[0,1]$ into M which send 0 to x_0 and which maps 1 to $M \backslash x_0$. Let $\mathrm{Hom}^*(\mathrm{SU}(2); G)$ denote the space of homomorphisms of SU(2) into G which generate $\pi_3(G)$. (Choose one such homomorphism, ρ_0, which defines a fiducial SU(2) subgroup of G and serves to identify the space of all such homomorphisms with G/G', where $G' \equiv \{g \in G\colon \mathrm{Ad}(g)|_{\mathrm{SU}(2)} = \mathrm{Identity}|_{\mathrm{SU}(2)}\}$.)

The space of homotopy equivalences $T\colon \mathscr{B}'(k,\eta) \to \mathscr{B}'(k + c(G), \eta)$ under consideration is parametrized by $\mathfrak{P}(x_0) \times \mathrm{Hom}^*(\mathrm{SU}(2); G) \times (0,1)$. Loosely speaking, this parametrization works as follows: Let $\mathrm{Fr}\, M$ denote the principal SO(4) bundle of orthonormal frames in T^*M. Think of a point $f \in \mathrm{Fr}\, M|_x$ as defining a Gaussian coordinate system on a neighborhood of x. Let $\gamma \in \mathfrak{P}(x_0)$ have end point x. The Levi-Civita connection plus a choice of fiducial frame, f_0, for $T^*M|_{x_0}$ defines, by parallel transport from x_0 along γ, a frame for $T^*M|_x$, and hence a Gaussian coordinate system on a ball about x. Likewise, a choice of a point $h \in P|_{x_0}$ and a connection, A, on P defines a point in $P|_x$. By parallel transport out along the radial geodesics through x, this data defines a trivialization of the bundle P over the coordinate ball which is centered at x.

On S^4, there is a fiducial, centered (in the sense of §3 of [22]) self-dual orbit $[A_1]$ on the principal SU(2) bundle which is defined by the fibration

$S^7 \to S^4$. A conformal scaling, $\lambda \in (0,1]$, and a choice of homomorphism, $\rho\colon \mathrm{SU}(2) \to G$, defines a new connection A' ($\equiv A + \lambda^* A_1$) on a bundle $P' \to M$ with characteristic classes $(k + c(G), \eta)$. The pair (P', A') is canonically isomorphic to (P, A) on the complement of a ball of radius $\mathscr{O}(\lambda)$ about x. Inside the ball of radius λ about x, the connection A' is canonically isomorphic to a conformal scaling of the self-dual connection A_1 from S^4.

The construction just outlined associates to each point $(\gamma, \rho, \lambda) \in \mathfrak{P}(x_0) \times \mathrm{Hom}^*(\mathrm{SU}(2); G) \times (0,1]$ a homotopy equivalence, $T[\gamma, \rho, \lambda]$, from $\mathscr{B}'(k,\eta)$ to $\mathscr{B}'(k + c(G), \eta)$. Since the parameter space $\mathfrak{P}(x_0) \times \mathrm{Hom}^*(\mathrm{SU}(2); G) \times (0,1)$ is connected, any pair of these homotopy equivalences are homotopic through homotopy equivalences.

It follows from calculations in [20] that for any fixed $b \in \mathscr{B}'(k, \eta)$,

$$\mathfrak{a}(T[\gamma, \rho, \lambda](b)) = \mathfrak{a}(b) + \mathscr{O}(\lambda^2). \tag{2.3}$$

Equation (2.3) implies that $\mathfrak{a}_{T\mathfrak{F}} \leq \mathfrak{a}_{\mathfrak{F}}$ for every homotopy invariant ($\mathrm{rel}\, \mathscr{B}'_\epsilon(k,\eta)$) family of compact subsets of $\mathscr{B}'(k,\eta)$.

The aforementioned calculations also show that for fixed b, it is possible to choose γ and ρ in such a way that when λ is sufficiently small,

$$\mathfrak{a}(T[\gamma, \rho, \lambda](b)) < \mathfrak{a}(b). \tag{2.4}$$

Unfortunately, there are obstructions to choosing (γ, ρ, λ) to vary continuously over a compact set $U \subset \mathscr{B}'(k,\eta)$ in such a way that (2.4) will hold for all points b in U.

Notice, however, that if (2.4) is obeyed for a given (γ, ρ, λ) and $b \in \mathscr{B}'(k,\eta)$, then it will hold for all b' in some small neighborhood of b. For a given compact U, one can find a finite set $\{(\gamma_i, \rho_i, \lambda_i)\}_{i \leq N}$ such that for any $b \in U$, there exist an index i for which (2.4) will hold for $(\gamma, \rho, \lambda) = (\gamma_i, \rho_i, \lambda_i)$. There is no loss of generality in requiring that the points $\{\gamma_i(1)\} \subset M$ be distinct, and that for $i \neq j$, $(\lambda_i + \lambda_j) < \mathrm{const} \cdot \mathrm{dist}(\gamma_i(1), \gamma_j(1))$.

Now, allow each λ_i to be a continuous function on $\mathscr{B}'(k,\eta)$ with values in $(0,1]$. To determine the behavior of the function λ_i, put $(\gamma, \rho, \lambda) = (\gamma_i, \rho_i, \lambda_i)$ in (2.3). When the correction term is positive for a given $b \in \mathscr{B}'(k,\eta)$, require that $\lambda_i(b)$ be very small. When the correction term in (2.3) is slightly negative, make $\lambda_i(b)$ relatively large.

By requiring the variation in the functions $\lambda_i(\cdot)$ as described above, a homotopy equivalence $T_U\colon \mathscr{B}'(k,\eta) \to \mathscr{B}'(k + c(G)\cdot N, \eta)$ has been constructed which has the property that the supremum of $\mathfrak{a}$ on the given compact set U is strictly greater than said supremum on $T(U)$.

A constant $z < 1$ can be derived from the group G and from the properties of the Riemannian metric on M with the following property: Choose $\epsilon > 0$.

Let $U \in \mathscr{B}'(k,\eta)$ be a compact set. With a little care (see §4 and Proposition 4.2), an integer $N < \infty$ and a homotopy equivalence $T_U : \mathscr{B}'(k,\eta) \to \mathscr{B}'(k+c(G)\cdot N,\eta)$ can be found with the property that T_U maps $\mathscr{B}'_\epsilon(k,\eta)$ into $\mathscr{B}'_{2\epsilon}(k+c(G)\cdot N,\eta)$. Also,

$$\sup_{T(U)} \mathfrak{a} < z \cdot \sup_{U} \mathfrak{a} + \epsilon. \tag{2.5}$$

This last equation has the following implication (Proposition 4.2): Given $\epsilon > 0$, and a homotopy invariant (rel $\mathscr{B}'_\epsilon(k,\eta)$) family of compact subsets of $\mathscr{B}'(k,\eta)$, $\mathfrak{F}$, there exists $N < \infty$, and there exists a homotopy equivalence $T: \mathscr{B}'(k,\eta) \to \mathscr{B}'(k+c(G)\cdot N,\eta)$ which maps $\mathscr{B}'_\epsilon(k,\eta)$ into $\mathscr{B}'_{2\epsilon}(k+c(G)\cdot N,\eta)$, and there exists a compact set $U \in T\mathfrak{F}$ which lies in $\mathscr{B}'_{2\epsilon}(k+c(G)\cdot N,\eta)$.

On S^4 with its standard metric, the facts in the preceding paragraph imply Theorems 1 and 2. Indeed, the pointwise positivity of a part of the Riemannian curvature tensor (as a section of $\operatorname{End}(P_- \bigwedge^2 T^*)$) can be used to construct $\epsilon_0 > 0$ (which is independent of k and η) together with a strong deformation retraction of $\mathscr{B}'_{\epsilon_0}(k,\eta)$ onto $\mathfrak{M}'(k,\eta)$. This deformation is a somewhat refined version of one which is constructed in [22]. (In [22], the $\epsilon > 0$ which defined the domain of the retraction was not independent of the Pontrjagin class of the bundle.)

In the general case, there is typically no retraction of all of $\mathscr{B}'_\epsilon(k,\eta)$ onto $\mathfrak{M}'(k,\eta)$ when $\epsilon > 0$. As in [21], the obstruction to constructing such a retraction can be thought of, locally, as a vector bundle over $\mathscr{B}'_\epsilon(k,\eta)$ which can be identified, over a small open set in $\mathscr{B}'_\epsilon(k,\eta)$, with the vector bundle whose fiber at $[A,h]$ is spanned by the L^2-eigenvectors of the operator $P_-d_A(P_-d_A)^*: C^\infty(P_-\Omega^2(\operatorname{Ad}P)) \to C^\infty(P_-\Omega^2(\operatorname{Ad}P))$ with small eigenvalues (see §§1–3 of [21]). Indeed, if $[A,h] \in \mathscr{B}'_\epsilon(k,\eta)$, and if $[A+a,h] \in \mathfrak{M}'(k,\eta)$, then $a \in C^\infty(\Omega^1(\operatorname{Ad}P))$ must obey the differential equation

$$P_-d_Aa + P_-(a \wedge a) + P_-F_A = 0. \tag{2.6}$$

The obstruction to inverting the operator

$$P_-d_A : C^\infty(\Omega^1(\operatorname{Ad}P)) \to C^\infty(P_-\Omega^2(\operatorname{Ad}P))$$

are precisely the elements in the kernel of $P_-d_A(P_-d_A)^*$. Since (2.6) is nonlinear, one finds all eigenvectors with small eigenvalue (small is determined by the L^2-norm of P_-F_A) to be obstructions.

The local vector bundle structure for these obstructions is a consequence of the fact that the L^2-eigenvalues of $P_-d_A(P_-d_A)^*$ vary continuously with the orbit $[A,h]$ (see §5). Thus, if $\mu \geq 0$ is not an eigenvalue of $P_-d_A(P_-d_A)^*$, then μ is not an eigenvalue of $P_-d_{A'}(P_-d_{A'})^*$ for any $[A',h']$ which is sufficiently close to $[A, h]$. For each $\mu \in (0, 1]$, one can define the open set $\mathfrak{U}(\mu) \equiv \{[A, h]: \mu \text{ is not an eigenvalue of } P_{-}d_A(P_{-}d_A)^*\}$. Over each such open set, one has the finite-dimensional vector bundle, $\mathfrak{V}(\mu)$, whose fiber over $[A, h] \in \mathfrak{U}(\mu)$ is the span of the eigenvectors of $P_{-}d_A(P_{-}d_A)^*$ with eigenvalue less than μ. If $\mu < \mu'$, then over $\mathfrak{U}(\mu) \cap \mathfrak{U}(\mu')$, there is the natural vector bundle inclusion $p(\mu, \mu'): \mathfrak{V}(\mu) \to \mathfrak{V}(\mu')$. For any $0 < \mu_1 < 1$, the family of open sets $\{\mathfrak{U}(\mu)\}_{\mu \in (\mu_1, 1)}$ form an open cover of $\mathscr{B}'(k, \eta)$.

The equations for self-duality naturally define ϵ_0, $\mu_1 > 0$, and, for $\mu \in (\mu_1, 1)$, a section $\mathfrak{s}_\mu$ of $\mathfrak{V}(\mu)$ over $\mathfrak{U}(\mu) \cap \mathscr{B}'_{\epsilon_0}(k, \eta)$. The numbers ϵ_0 and μ_1 are independent of (k, η). The construction of $\mathfrak{s}_\mu$ is described in §5. The set $\{\mathfrak{s}_\mu\}_{\mu \in (\mu_1, 1)}$ has the property that over $\mathfrak{U}(\mu) \cap \mathfrak{U}(\mu')$, $\mathfrak{s}_{\mu'} = p(\mu, \mu') \cdot \mathfrak{s}_\mu$. Furthermore, if $[A, h] \in \mathfrak{U}(\mu) \cap \mathfrak{M}'(k, \eta)$, then $\mathfrak{s}_\mu([A, h]) = 0$.

From the $\{\mathfrak{s}_\mu\}_{\mu \in (\mu_1, 1)}$, a smooth homotopy of $\mathscr{B}'(k, \eta)$ is constructed in §5 which maps $\mathscr{B}'_{\epsilon_0}(k, \eta)$ into $\mathscr{B}'_{z \cdot \epsilon_0}(k, \eta)$ for fixed $z < \infty$, which fixes $\mathfrak{M}'(k, \eta)$, and which homotopes $\bigcup_{\mu \in (\mu_1, 1)} \mathfrak{s}_\mu^{-1}(0)$ onto $\mathfrak{M}'(k, \eta)$.

The formal codimension of a connected component of $\mathfrak{s}_\mu^{-1}(0) \cap \mathscr{B}'_{\epsilon_0}(k, \eta)$ is equal to the dimension, for fixed $[A, h]$ in that component, of the span of the L^2-eigenvectors of $P_{-}d_A(P_{-}d_A)^*$ with eigenvalue less than μ. This dimension is bounded a priori, given the group G and the Riemannian metric; a fact which indicates that the inclusion of $\mathfrak{s}_\mu^{-1}(0) \cap \mathscr{B}'_{\epsilon_0}(k, \eta)$ into $\mathscr{B}'_{\epsilon_0}(k, \eta)$ is responsible for more and more of the topology of $\mathscr{B}'_{\epsilon_0}(k, \eta)$.

Such an a priori dimension bound is established in the Appendix. There, operators of the form

$$\nabla^*\nabla + R\colon C^\infty \text{ (Hermitian vector bundle)} \to C^\infty \text{ (Hermitian vector bundle)}$$

are considered when ∇ is a metric compatible connection, and R is a section of the associated bundle of endomorphisms. The principle result in the Appendix is a proof of the assertion that the dimension of the span of the L^2-eigenvectors $\nabla^*\nabla + R$ with eigenvalue less than some $E < \infty$ is bounded a priori by a function of the Riemannian metric, of E and of the L^2-norm of R. A similar result was announced by Berard and Besson [4] using a theorem of Cwickel-Lieb-Rosenbljum [18].

Let $b \equiv [A, h] \in \mathscr{B}'_{\epsilon_0}(k, \eta)$ be an orbit which is not in $\mathfrak{s}_\mu^{-1}(0)$. (One knows a priori that $\|\mathfrak{s}_\mu(b)\|_{L^2}$ is $\mathscr{O}(\epsilon_0)$.) Fix a point $(\gamma, \rho, \lambda) \in \mathfrak{P}(x_0) \times \text{Hom}^*(\text{SU}(2); G) \times (0, 1)$. As discussed, such a point determines a homotopy equivalence $T[\gamma, \rho, \lambda]: \mathscr{B}'(k, \eta) \to \mathscr{B}'(k + c(G), \eta)$. For small λ, one can compare $\mathfrak{s}_\mu(T[\gamma, \rho, \lambda](A, h))$ with $\mathfrak{s}_\mu([A, h])$. The calculations generalize calculations in [21].

The calculations are tractable for the following reason: For a large open set of $\mu \in (\mu, 1)$, the homotopy equivalence $T[\gamma, \rho, \lambda]$ defines an isomorphism, which is an isometry to $\mathscr{O}(\lambda^2)$, between the vector space $\mathfrak{V}(\mu)|_{[A,h]}$ and the vector space $\mathfrak{V}(\mu)|_{T[\gamma,\rho,\lambda](A,h)}$. The isomorphism in question is constructed by exploiting the two facts. The first is the fact that the orbits $[A, h]$ and $T[\gamma, \rho, \lambda](A, h)$ have a canonical identification on the exterior of a ball of radius λ about $\gamma(1)$. The second is the fact that the L^2-eigenvectors of $P_-d_A(P_-d_A)^*$ with eigenvalue less than 1 obey (k, η)-independent a priori estimates when $[A, h] \in \mathscr{B}'_{\epsilon_0}(k, \eta)$. So, a linear map from $\mathfrak{V}(\mu)|_{[A,h]}$ to $\mathfrak{V}(\mu)|_{T[\gamma,\rho,\lambda](A,h)}$ is obtained by first deforming a given eigenvector (section of $P_-\Omega^2(\operatorname{Ad} P)$) to be zero near $\gamma(1)$. Second, identifying the vector bundles $P_-\Omega^2(\operatorname{Ad} P)$ and $P_-\Omega^2(\operatorname{Ad} P')$ in the complement of $\gamma(1)$. Finally, project onto $\mathfrak{V}(\mu)|_{T[\gamma,\rho,\lambda](A,h)}$ with the L^2-orthogonal projection (see §6).

The calculation yields the identity

$$\mathfrak{s}_\mu(T[\gamma, \rho, \lambda](A, h)) = T[\gamma, \rho, \lambda] \cdot (\mathfrak{s}_\mu([A, h]) + \lambda^2 \cdot r(\gamma, \rho; [A, h])^* + \lambda^{5/2} \cdot e(\gamma, \rho, \lambda, [A, h])), \tag{2.7}$$

where $r(\gamma, \rho; [A, h])^*$ is the adjoint of a linear functional, $r(\gamma, \rho; [A, h])\colon \mathfrak{V}(\mu)|_{[A,h]} \to \mathbf{R}$ which is defined as follows: The choice of a fiducial frame f_0 for $T^*M|_{x_0}$ identifies $P_-T^*M|_{x_0}$ with $\mathfrak{su}(2)$, the Lie algebra of $\mathrm{SU}(2)$. Similarly, the choice of $h \in P|_{x_0}$ identifies $\operatorname{Ad} P|_{x_0}$ with $\mathfrak{g}$. A unit length vector $\omega(\rho)$ in $P_-\Omega^2(\operatorname{Ad} P)|_{x_0}$ is then defined by the homomorphism ρ. Parallel transport of $\omega(\rho)$ along γ via the Levi-Civita connection and the connection A defines a unit vector in $P_-\Omega^2(\operatorname{Ad} P)|_{\gamma(1)}$. Finally, the linear functional r is obtained by evaluating an eigenvector at $\gamma(1)$ and then contracting with this unit vector. The $\lambda^{5/2} \cdot e(\cdot)$ term in (2.7) is a uniformly bounded correction term.

Let $\{\tau_a\}_{a=1}{}^3$ be an orthonormal basis for $\mathfrak{su}(2)$. A linear algebra argument shows that m-points $\{\rho_i\} \in \operatorname{Hom}^*(\mathrm{SU}(2); G)$ can be found so that the map $\Upsilon\colon (0, \infty)^m \to \mathfrak{g} \otimes \mathfrak{su}(2)$ which assigns to $t \equiv (t_1, \cdots, t_m)$

$$\Upsilon(t) \equiv \sum_{j=1}^{m} \sum_{a=1}^{3} t_j \cdot (\rho_j * (\tau_a) \otimes \tau_a) \tag{2.8}$$

is a surjection from a compact set onto a neighborhood of 0 in $\mathfrak{g} \otimes \mathfrak{su}(2)$.

This last fact, plus the uniform a priori estimates on the eigenvectors which span $\mathfrak{V}(\mu)$, and the uniform bound on their number allow the construction, from a compact set $U \subset \mathscr{B}'_{\epsilon_0}(k, \eta)$, of $N < \infty$ points $\{(\gamma_i, \rho_i)\}$ and N smooth functions $\{\lambda_i(\cdot)\colon \mathscr{B}'(k, \eta) \to (0, 1)\}$ with the following properties: First, the endpoints $\{\gamma_i(1)\}$ are distinct, and $|\lambda_i(\cdot) + \lambda_j(\cdot)| < \operatorname{dist}(\gamma_i(1), \gamma_j(1))$ for $i \neq j$. Second, for each $b \in U$ and for each $\mu \in (\mu_1, 1)$,

$$0 = \mathfrak{s}_\mu(b) + \sum_{j=1}^{N} (\lambda_j(b)^2 \cdot r(\gamma_{j,\rho_j;b})^* + \lambda_j(b)^{5/2} \cdot e_j(\gamma_j, \rho_j, \lambda_j(b), b)). \tag{2.9}$$

Third, the data $\{(\gamma_i, \rho, \lambda_i(\cdot))\}$ defines a homotopy equivalence $T\colon \mathscr{B}'(k,\eta) \to \mathscr{B}'(k+c(G)\cdot N, \eta)$ which maps $\mathscr{B}'_{\epsilon_0}(k,\eta)$ into $\mathscr{B}'_{z\epsilon_0}(k,\eta)$ with z independent of U and (k,η). These observations are made in §6.

Due to (2.9), the map T sends U into $\bigcup_{\mu\in(\mu_1,1)} \mathfrak{s}_\mu^{-1}(0)$. This last observation essentially completes the proofs of Theorems 1 and 2. The final arguments are given in §7.

The next section provides a short summary of background facts which will be assumed, and an introduction to the notation which will be used in this article.

3. Preliminaries

A proper beginning introduces the minimal notation and background material which a reasonable reader might require. This section presumes to serve; the reader is also referred to [22] and to [13].

It is convenient to consider the affine space $\mathscr{A}(P)$ of smooth connections on a principal G-bundle $P \to M$ as a dense subset of the affine Banach manifold, $\mathfrak{A}(P)$, of L^2_2-connections on P [13]. The gauge group $\mathscr{G}(P)$ of smooth automorphisms of P is a dense subgroup of the Banach Lie group, $\mathfrak{G}(P)$, of L^2_3 automorphisms of P. The group $\mathfrak{G}(P)$ contains the pointed L^2_3-gauge group $\mathfrak{G}_0(P) = \{g \in \mathfrak{G}\colon g(s) = 1\}$. Suppose that P has characteristic classes (k,η). Then, as described in [13], $\mathfrak{B}'(k,\eta) \equiv \mathfrak{B}'(P) = \mathfrak{A}(P)/\mathfrak{G}_0(P)$ is a smooth Banach manifold with an L^2_2-Sobolev space for its model. Alternately, fixing a base point $x_0 \in M$ identifies $\mathfrak{B}'(k,\eta)$ with $(\mathfrak{A}(P) \times P|_{x_0})/\mathfrak{G}(P)$. The projection $\pi\colon \mathfrak{A}(P) \to \mathfrak{B}'(k,\eta)$ defines a principal $\mathfrak{G}_0(P)$ bundle.

Let $\mathfrak{B}(k,\eta) \equiv \mathfrak{B}(P) = \mathfrak{A}(P)/\mathfrak{G}(P)$ be defined as a topological space with the quotient topology. It is not quite a Banach manifold; but denote by $\mathfrak{R}(P)$ the infinite codimensional set of reducible connections on P and let $\mathfrak{A}^\sharp(P) \equiv \mathfrak{A}(P)\backslash\mathfrak{R}(P)$. Then, $\mathfrak{B}^\sharp = \mathfrak{A}^\sharp/\mathfrak{G}$ is a smooth Banach manifold and the quotient map $\pi\colon \mathfrak{A} \to \mathfrak{B}$ defines a smooth principal $\mathfrak{G}/\mathrm{Center}\,\mathfrak{G}$-bundle over $\mathfrak{B}^\sharp$, and the projection $\mathfrak{B}'(k,\eta) \to \mathfrak{B}$ defines a principal $G/\mathrm{Center}\,G$ bundle over $\mathfrak{B}^\sharp$.

In most of this article, the distinction between $\mathfrak{A}$ and $\mathscr{A}$, $\mathfrak{B}'$ and $\mathscr{B}'$, or $\mathfrak{B}$ and $\mathscr{B}$ will be irrelevant to the arguments, so the spaces will generally not be distinguished.

For $q \in \{0, 1, \cdots, 4\}$, let $\Omega^q(\mathrm{Ad}\,P)$ denote the vector bundle $\mathrm{Ad}\,P \otimes \bigwedge^q T^*$. Fix a smooth connection A_0 on P. With $A = A_0$, one defines the L^p_k-Sobolev norm on $\Omega^q(\operatorname{Ad} P)$ as follows: For a section ψ set

$$\|\psi\|_{p,k}^p = \int_M \sum_{j=0}^{k} |\nabla_A^{(j)}\psi|^p, \tag{3.1}$$

where ∇_A is the covariant derivative from the connection A on P and from the given Riemannian metric's Levi-Civita connection on tensor bundles that are constructed from the tangent and cotangent bundles of M. In (3.1), $\nabla_A^{(j)} = \nabla_A \dots \nabla_A$, a total of j-times. The norm above is defined with the metric on $\operatorname{Ad} P$ which comes from the Killing metric on $\mathfrak{g}$ and with the metric on the tensor bundles which is induced by the standard Riemannian metric on the tangent bundle of M.

Let $L_k^p(\Omega^q(\operatorname{Ad} P))$ denote the Banach space which is obtained by completing the space of smooth sections of $\Omega^q(\operatorname{Ad} P)$ in the norm of (3.1). (A different choice of smooth connection in $\mathfrak{A}(P)$, or a different choice of metric with which to define the norm in (3.1), will yield the same Banach space but with an equivalent norm.) Thus, for smooth A, $\nabla_A\colon L_k^2(\Omega^q(\operatorname{Ad} P)) \to L_{k-1}^2(\Omega^q(\operatorname{Ad} P) \otimes T^*M)$ is a bounded operator. For $A \in \mathfrak{A}(P)$, this is only true for $k = 1, 2, 3$.

The choice of metric on TM also allows for the definition of the formal L^2 adjoint of ∇_A, the operator ∇_A^* which sends $L_k^2(\Omega^q(\operatorname{Ad} P) \otimes T^*M)$ to $L_{k-1}^2(\Omega^q(\operatorname{Ad} P))$. For $A \in \mathfrak{A}(P)$ such is the case only for $k = 1, 2, 3$.

The choice of a smooth connection $A \in \mathfrak{A}(P)$ defines the covariant exterior derivative $d_A\colon L_k^2(\Omega^q(\operatorname{Ad} P)) \to L_{k-1}^2(\Omega^{q+1}(\operatorname{Ad} P))$. The fixed metric on TM allows the definition of the formal L^2 adjoint of d_A, this is the operator $d_A^*\colon L_k^2(\Omega^q(\operatorname{Ad} P)) \to L_{k-1}^2(\Omega^{q-1}(\operatorname{Ad} P))$. Again, if $A \in \mathfrak{A}(P)$, then such is the case only for $k = 1, 2, 3$.

The metric on TM defines the self-dual, P_+, and anti-self-dual, P_-, projections on $\Omega^2(\operatorname{Ad} P)$. These decompose $\Omega^2(\operatorname{Ad} P)$ as $P_+\Omega^2(\operatorname{Ad} P) \otimes P_-\Omega^2(\operatorname{Ad} P)$.

For a connection A in $\mathfrak{A}(P)$, its curvature F_A is in $L_1^2(\Omega^2(\operatorname{Ad} P))$. Thus, the functional $\mathfrak{a}$ in (2.1) is finite on $\mathfrak{B}(k, \eta)$, and one can check easily that it is a smooth functional on $\mathfrak{B}'(k, \eta)$ and on $\mathfrak{B}^\sharp(k, \eta)$.

In studying the geometry of $\mathfrak{B}'(k, \eta)$ or $\mathfrak{B}(k, \eta)$, it is convenient to introduce various infinite dimensional vector bundles over $\mathfrak{B}'(k, \eta)$ whose fibers are Banach spaces of sections of $\operatorname{Ad} P$-valued differential forms. Define $\mathfrak{V}^p \to \mathfrak{B}'(k, \eta)$ to be the vector bundle $(\mathfrak{A}(P) \times P|_{x_0} \times L_2^2(\Omega^p(\operatorname{Ad} P)))/\mathfrak{G}(P)$. One also defines vector bundles $(\mathfrak{A}(P) \times P|_{x_0} \times L_2^2(P_\pm\Omega^2(\operatorname{Ad} P)))/G(P)$ over $\mathfrak{B}'(k, \eta)$. Note that these vector bundles are the pull-backs of the obvious vector bundles over $\mathfrak{B}^\sharp(k, \eta)$.

The vector bundle $\mathfrak{V}^p$ has a convenient fiber metric: Let $u = [A, h, v]$ and $v = [A, h, \psi]$ be a points in $\mathfrak{V}^p$ over $[A]$ in $\mathfrak{B}'(k, \eta)$. Then, set

$$(3.2) \qquad \langle u, v\rangle_{[A]} = \int_M \{(\nabla_A \psi, \nabla_A v) + (\psi, v)\}.$$

The assignment of $b \in \mathfrak{G}^\sharp$ to $\langle\cdot,\cdot\rangle_b$ defines a smoothly varying metric and associated norm $(\|\cdot\|_b)$ on the vector bundle $\mathfrak{V}^p$. This fact is a straightforward application of the 4-dimensional Sobolev inclusion $L^2_1 \to L^4$ together with Kato's inequality. Recall that Kato's inequality asserts that for a $L^2_{1,\mathrm{loc}}$-connection, A, and for an $L^2_{1,\mathrm{loc}} - \operatorname{Ad} P$ valued differential form, ψ, one has

$$(3.3) \qquad |\nabla_A \psi| \geq |d|\psi|\,| \quad \text{almost everywhere.}$$

This inequality implies (see (2.14) in [20])

$$(3.4) \quad \|\psi\|^2_A \equiv \|\nabla_A \psi\|^2_{L^2} + \|\psi\|^2_{L^2} \geq z_0 \cdot \left(\|\psi\|^2_{L^4} + \sup_{x \in M} \|\operatorname{dist}(\cdot, x)^{-1} \cdot \psi\|^2_{L^2}\right),$$

with z_0 depending only on the Riemannian metric on TM.

The affine structure of the space of connections induces a smooth map $f\colon \mathfrak{V}^1 \to \mathfrak{B}'(k, \eta)$ which is the canonical projection when restricted to the canonical zero section of $\mathfrak{V}^1$. This map sends $v = [A, h, \psi]$ to $f(v) = [A + \psi, h]$.

One final piece of notation: An L^2_2-connection on a principal bundle $P \to S^4$ is called *centered* when

$$(3.5) \qquad \int_{\mathbf{R}^4} (y^\nu, 1 - |y|^2) \cdot |\Phi^* F_A|^2(y) \cdot d^4 y = 0 \in \mathbf{R}^5,$$

where $\Phi\colon \mathbf{R}^4 \to S^4 \backslash \{\text{south pole}\}$ is the inverse to a stereographic projection (see §3 of [22]).

4. The topology of $\mathfrak{B}'(k, \eta)$ for large k

The main purpose of this section is to analyze how the topology in $\mathfrak{B}'(k', \eta)$ behaves as k increases; specifically, this section contains the proofs of Propositions 4.1 and 4.2 below.

Proposition 4.1. *Let G be a compact, simple Lie group. Then, there exists an integer $c(G) > 0$ with the following significance: Let $(k, \eta) \in \mathbf{Z} \times H^2(M; \pi_1(G))$ be admissible as characteristic classes for a principal G-bundle over M. Then $(k \pm c(G), \eta)$ is admissible as characteristic classes for a principal G-bundle over M, and there is a homotopy equivalence between $\mathfrak{B}'(k, \eta)$ and $\mathfrak{B}'(k + c(G), \eta)$. Given a smooth function $\mathfrak{f}\colon \mathfrak{B}'(k, \eta) \to (0, \infty)$, there is a homotopy equivalence $\theta\colon \mathfrak{B}'(k, \eta) \to \mathfrak{B}'(k + c(G), \eta)$ which has the property that $\mathfrak{a}(\theta(b)) < \mathfrak{a}(b) + \mathfrak{f}(b)$ for any $b \in \mathfrak{B}'(k, \eta)$.*

Since the space of relative homotopy equivalences between $(\mathfrak{B}'(k,\eta), \mathfrak{B}'_\delta(k,\eta))$ and $(\mathfrak{B}'(k+c(G),\eta), \mathfrak{B}'_{z\cdot\delta}(k+c(G),\eta))$ may not be connected, one must be careful when identifying the relative topology in $(\mathfrak{B}'(k,\eta), \mathfrak{B}'_\delta(k,\eta))$ with that in $(\mathfrak{B}'(k+c(G),\eta), B'_{z\cdot\delta}(k+c(G),\eta))$. Such an identification can be made once a homotopy class of homotopy equivalences is specified. Suppose that $\mathfrak{F}$ is a homotopy invariant family of compact subsets of $\mathfrak{B}'(k,\eta)$ (rel $\mathfrak{B}'_\delta(k,\eta)$). Let $T\colon \mathfrak{B}'(k,\eta) \to \mathfrak{B}'(k+c(G),\eta)$ be a homotopy equivalence which maps $\mathfrak{B}'_\delta(k,\eta)$ into $\mathfrak{B}'_{z\cdot\delta}(k+c(G),\eta))$. Then no ambiguity arises by defining $T\mathfrak{F}$ to be that homotopy invariant (rel $\mathfrak{B}'_{z\cdot\delta}(k+c(G),\eta)$) family of compact subsets of $\mathfrak{B}'(k+c(G),\eta)$ which consists of the compact sets U of the form $\Phi(1,T(V))$, where $V \in \mathfrak{F}$, and where $\Phi(\cdot,\cdot)\colon [0,1]\times\mathfrak{B}'(k+c(G),\eta) \to \mathfrak{B}'(k+c(G),\eta)$ is a continuous homotopy (rel $\mathfrak{B}'_{z\cdot\delta}(k+c(G),\eta)$).

It will be implicit in this article that any equivalence between $\mathfrak{B}'(k,\eta) \to \mathfrak{B}'(k+c(G),\eta)$ will be chosen from a specific, connected space, $\mathfrak{T}$, of homotopy equivalences. Any such equivalence which maps $\mathfrak{B}'_\delta(k,\eta)$ into $\mathfrak{B}'_{z\cdot\delta}(k+c(G),\eta))$ will be chosen from a connected subspace, $\mathfrak{T}(\delta, z\cdot\delta)$, of homotopy equivalences which is connected as a space of homotopy equivalences which map $\mathfrak{B}'_\delta(k,\eta)$ into $\mathfrak{B}'_{2z\cdot\delta}(k+c(G),\eta))$. With this understood, there is no ambiguity in writing $\mathfrak{F}(1)$ for $T\mathfrak{F}$ when $T\in\mathfrak{T}$ (or $\mathfrak{T}(\delta, z\cdot\delta)$), and when $\mathfrak{F}$ is a homotopy invariant family of compact subsets of $\mathfrak{B}'(k,\eta)$ (rel $\mathfrak{B}'_\delta(k,\eta)$). A specific, connected set, $\mathfrak{T}_0$, of homotopy equivalences between $\mathfrak{B}'(k,\eta)$ and $\mathfrak{B}'(k+c(G),\eta)$ is constructed for the proof of Proposition 4.1; and the space $\mathfrak{T}$ is the space of all maps from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G),\eta)$ which are homotopic to those in $\mathfrak{T}_0$.

Proposition 4.2. *Let (k_0,η) be admissible characteristic classes for a principal G-bundle over M. Fix $\epsilon > 0$. Let $\mathfrak{F}$ denote a homotopy invariant family of compact subsets of $\mathfrak{B}'(k_0,\eta)$ (rel $\mathfrak{B}'_\epsilon(k,\eta)$), and for each integer $j > 0$, let $\mathfrak{F}(j) \equiv T_j(T_{j-1}(\ldots T_1(\mathfrak{F})\ldots))$ denote the homotopy equivalent, homotopy invariant family of compact subsets of $\mathfrak{B}'(k_0+c(G)\cdot j,\eta)$ (rel $\mathfrak{B}'_{2\epsilon}(k+c(G)\cdot j,\eta)$). Here $T_k\colon \mathfrak{B}'(k_0+c(G)\cdot(k-1),\eta) \to B'(k_0+c(G)\cdot k,\eta)$ is a homotopy equivalence in $\mathfrak{T}((1+(k-1)/j)\cdot\epsilon,\ (1+k/j)\cdot\epsilon)$. For each j, define $\mathfrak{a}_{\mathfrak{F}(j)}$ by (2.2). Then $\mathfrak{a}_{\mathfrak{F}(j+1)} \le \mathfrak{a}_{\mathfrak{F}(j)}$ and $\lim_{j\to\infty}\mathfrak{a}_{\mathfrak{F}(j)} \le 2\cdot\epsilon$.*

Proof of Proposition 4.1. Let $P_0 \to M$ be a principal G-bundle with characteristic classes (k,η). Let $P_\pm \to S^4$ be a principal SU(2)-bundle with characteristic number $k = \pm 4$. (In fact, $P_\pm \to S^4$ is topologically the Hopf fibration $S^7 \to S^4$. Let $\mathbf{H}$ denote the quaternion algebra, and think of S^7 as the unit sphere in $\mathbf{H}^2$. Think of SU(2) as the group of quaternions with norm 1. Then SU(2) acts freely on S^7 by the right or the left action; depending upon one's convention, one action gives P_+ and the other gives P_-.) Let $i\colon \mathrm{SU}(2) \to G$ be a group homomorphism which generates $\pi_3(G)$. Let $P_{G_\pm} \equiv P_\pm \times_i G$. By gluing $P_{G_\pm}$ to P_0 as specified in §4 of [22] or §6 in [20], one obtains a principal G-bundle with characteristic class $(k \pm c(G), \eta)$, where

$$c(G) \equiv 4 \times \sup_{\sigma \in \mathfrak{su}(2)} (i * \sigma, i * \sigma)_G / (\sigma, \sigma)_{\mathrm{SU}(2)},$$

and $(\cdot,\cdot)_G$ is the Casimir inner product on $\mathfrak{g}$ (see [2]). This proves the first assertion of the proposition.

Let $[A_1 \equiv A_\pm] \in \mathfrak{B}(P_\pm)$ denote the unique orbit of a centered (in the sense of (3.5)), self-dual connection on P_+ or anti-self-dual connection on P_-; here, self-dual is with respect to the standard metric on S^4. (See [2] for a proof of the uniqueness of this orbit.) In a natural way, $[A_\pm]$ defines an orbit in $\mathfrak{B}(P_{G_\pm})$ since there are natural maps of pairs $(\mathfrak{A}(P_\pm), \mathfrak{G}(P_\pm)) \to (\mathfrak{A}(P_{G_\pm}), \mathfrak{G}(P_{G_\pm}))$ which commute with the group actions. Choose a point $[A_1, h_1]$ in $(\mathfrak{A}(P_{G_\pm}) \times P_{G_\pm}|_s)/\mathfrak{G}(P_{G_\pm})$ over $[A_1]$. Here, $s \equiv$ south pole.

Let $x_0 \in M$ be the base point. Fix a frame $f_0 \in \mathrm{Fr}\, M|_{x_0}$ and a smooth path $\varphi \in \mathfrak{P}(x_0)$ (as defined in §2). Choosing smooth function $t(\cdot)\colon \mathfrak{B}'(k,\eta) \to (0, 1/8]$ and $s(\cdot)\colon \mathfrak{B}'(k,\eta) \to (0,1]$ allows the definition of a map

$$\theta_\pm \equiv \theta(t,s)_\pm \colon \mathfrak{B}'(k,\eta) \to \mathfrak{B}'(k \pm c(G), \eta). \tag{4.1}$$

The map $\theta_\pm$ is defined to send $[A_0, h_0] \in \mathfrak{B}'(k,\eta)$ to $[A(w), h_0]$, where $A(w)$ is a connection on a bundle $P(w) \to M$ with characteristic classes $(k \pm c(G), \eta)$. Here, $w \equiv [[A_0, h_0], f_0, \varphi, s([A_0,h_0]), t([A_0,h_0]), [A_1, h_1]]$.

The bundle $P(w)$ and the connection $A(w)$ are defined as follows: Parallel transport f_0 along φ using the Levi-Civita connection to define a frame f in $\mathrm{Fr}\, M|_x$ where $x \equiv \varphi(1)$. The frame f defines a Gaussian coordinate system on a ball B in M which is centered at x; use this coordinate system to identify the ball with its image in $\mathbf{R}^4$. Parallel transport h_0 along φ using the connection A_0 to define a point h in $P|_x$. Then, parallel transport h along the radial geodesics through x to define a section $\phi(A_0, h_0, \varphi)$ of $P|_B$. This section identifies $P|_B$ with $B \times G$.

The inverse to stereographic projection, $\Phi\colon \mathbf{R}^4 \to S^4 \backslash s$, defines a conformal diffeomorphism. Use this to pull A_1 back to $\mathbf{R}^4$ as a self-dual connection on $\Phi^* P_{G_\pm}$. A positive number, λ, defines a conformal diffeomorphism of $\mathbf{R}^4$ by pulling back the coordinate functions, y, to $\lambda^* y = y/\lambda$. For $\lambda > 0$, $\lambda^*\Phi^* A_1$ is a self-dual connection on $\lambda^*\Phi^* P_{G_\pm}$.

Let $\lambda \equiv s(A_0, h_0) \cdot t(A_0, h_0)$. By parallel transport of h_1 by the connection $\lambda^*\Phi^* A_1$ along the radial geodesics through the south pole on S^4, a section $\phi(A_1, h_1, \lambda)$ of $\lambda^*\Phi^* P_{G_\pm}|_{\mathbf{R}^4\backslash 0}$ is defined; and thus, an identification of $\lambda^*\Phi^* P_{G_\pm}|_{\mathbf{R}^4\backslash 0}$ with $\mathbf{R}^4\backslash 0 \times G$ is made.

Let $\beta(\cdot) \in C^\infty([0,\infty),[0,1])$ obey $\beta(r) \equiv 1$ if $r < 1/2$, and $\beta(r) \equiv 0$ if $r > 1$. For $\rho > 0$ and $y \in M$, set $\beta_\rho(y) \equiv \beta(\mathrm{dist}(y,x)/\rho)$.

Define $(P(w), A(w))$ as follows: If $\operatorname{dist}(x, \cdot) < \lambda/2$, identify $P(w)$ with $\lambda^*\Phi^*P_{G_\pm}|_{\mathbf{R}^4}$ using the Gaussian coordinate system, and set

$$A(w; t) \equiv \lambda^*\Phi^*A_1. \tag{4.2a}$$

If $\operatorname{dist}(x, \cdot) > \lambda/t$, identify $P(w)$ with P and set

$$A(w, t) \equiv A_0. \tag{4.2b}$$

If $\lambda/2 \leq \operatorname{dist}(x, \cdot) \leq \lambda/t$, identify $P(w)$ with the trivial product bundle, and set

$$A(w, t) \equiv \Gamma + (1 - \beta_{8\lambda}) \cdot \phi(A_0, h_0, \varphi)^* A_0 + \beta_{\lambda/t} \cdot \phi(A_1, h_1, \lambda)^* \lambda^* \Phi^* A_1, \tag{4.2c}$$

where Γ is the product connection.

It is a straightforward task to check that the assignment of $[A_0, h_0]$ to $[A(w), h_0]$ defines a smooth map from $\mathscr{B}'(k, \eta) \to \mathscr{B}'(k \pm c(G), \eta)$. However, the map is continuous from $\mathfrak{B}'(k, \eta)$ to $\mathfrak{B}'(k \pm c(G), \eta)$ only if the former has an L^2_m-topology and the latter has an L^2_{m-1}-topology.

By specifying a mollifier, one can smooth the section $\phi(A_0, h_0, \varphi)$ over B but still approximate $\phi(A_0, h_0, \varphi)$ to any desired order in the L^2_m-topology. The closeness of the approximation can be allowed to depend on $[A_0, h_0]$; for the second assertion of Proposition 4.1, such an $[A_0, h_0]$-dependent smoothing is required. (See below for an extended discussion on a different approach to smoothing connections.)

After smoothing $\phi(\cdot, \varphi)$, a smooth map is obtained from $\mathfrak{B}'(k, \eta) \to \mathfrak{B}'(k \pm c(G), \eta)$ when both domain and range have the same L^2_m-topology.

A slight modification of the arguments in §6 of [20] proves that the compositions $\theta_+ \circ \theta_-$ and $\theta_- \circ \theta_+$ are both homotopic to the identity. This fact implies the first assertion of Proposition 4.1.

By choosing the $[A_0, h_0]$ dependence of the functions $s(\cdot)$ and $t(\cdot)$ and the mollifier appropriately, one can obtain a homotopy equivalence which accomplishes the requirements of the second assertion of Proposition 4.1. This fact is implied by Proposition 6.1 in [20].

The proof of Proposition 4.2 requires finding sets in $\mathfrak{F}$ which obey a priori estimates. For this reason, the following digression on smoothing of connections is required.

A given connection will be smoothed on a ball $B(x, r) \subset M$ of radius r and center x; the connection is smoothed by smoothing the connection 1-form as defined with respect to a particular trivialization of the bundle over the ball. This can be accomplished with a mollifier, as in the proof of Proposition 4.1, but for the proof of Proposition 4.2 it will be accomplished by replacing the connection 1-form by the solution of an elliptic, differential equation. The elliptic equation will provide the necessary a priori estimates. The equation in question is the variational equation of a smooth functional on the Hilbert space $H(x,r) \equiv \{v \in L_1^2(\Omega^1(\operatorname{Ad} P)|_{B(x,r)}) \colon v|_{\partial B(x,r)} \equiv 0\}$.

Define the Banach space

$$K_\epsilon(x,r) \equiv \{a \in L_1^2(T^*B(x,r) \times g \colon \|\nabla_\Gamma a\|^2_{L^2;B(x,r)} + \|a\|^2_{L^2;B(x,r)} \leq \epsilon \text{ and which satisfy } d_\Gamma^* a = 0 \text{ and } i^*_{\partial B(x,r)}(*a) = 0\}. \tag{4.3}$$

Here, Γ is the product connection. For each $a \in K_\epsilon(x,r)$, a smooth functional on $H(x,r)$ is defined by sending $v \in H(x,r)$ to

$$\mathfrak{s}_a(v) \equiv \int_{B(x,r)} (|P_- F_{\Gamma+a+v}|^2 + |d_\Gamma^* v|^2) \cdot d\,\mathrm{vol}. \tag{4.4}$$

Lemma 4.3. *Let M be a compact, Riemannian manifold. There exist r_0, $\epsilon \in (0,1]$ and $z < \infty$ with the following significance: Let $x \in M$, and let $r < r_0$. For each $a \in K_\epsilon(x,r)$, define the functional $\mathfrak{s}_a(\cdot)$ on $H(x,r)$ by (4.4). This functional has a critical point, $v_0(a)$, which is an absolute minimum, and is unique in having the following properties:*

(1) *For all*

$$t \in [0,1], \ \|P_- F_{\Gamma+a+tv_0}\|^2_{L^2;B(x,r)} \leq \mathfrak{s}_a(tv_0) \leq \|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r)},$$

and the right-hand inequality is an equality for $t \in (0,1]$ if and only if $v_0 \equiv 0$.

(2) $\|\nabla_\Gamma v_0\|^2_{L^2;B(x,r)} + \|v_0\|^2_{L^2;B(x,r)} \leq z \cdot \|\nabla \mathfrak{s}_a|_{v=0}\|^2_{L_1^{2*};B(x,r)}$.

(3) *If* $\|P_- F_{\Gamma+a+v_0}\|^2_{L^2;B(x,r/2)} \leq \frac{1}{2}\|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r/2)}$, *then*

$$\|P_- F_{\Gamma+a+v_0}\|^2_{L^2;B(x,r)} \leq (1 + z \cdot \epsilon^2) \cdot \|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r)} - \tfrac{1}{16}\|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r/2)}.$$

(4) *The $\mathfrak{g}$-valued 1-form $a + v_0(a)$ is smooth in the interior of $B(x,r)$, and for each $m \geq 0$, there exists $\xi_m < \infty$ which is independent of A, r, and x and is such that when $y \in B(x, 3r/4)$, then*

$$|\nabla_\Gamma^{(m)}(a + v(a))|(y) \leq \xi_m \cdot r^{-m-2} \cdot \|a\|_{L_1^2;B(x,r)}.$$

(5) *When $y \in B(x, 3r/4)$, then*

$$|P_- F_{\Gamma+a+v(a)}|(y) + r \cdot |\nabla_\Gamma P_- F_{\Gamma+a+v(a)}|(y) \leq \xi_0 \cdot r^{-2}\|P_- F_{a+v(a)}\|_{L^2;B(x,r)}.$$

(6) *The assignment of a 1-form a to $v_0(a) \in L_1^2(\Omega^1(\operatorname{Ad} P))$ defines a smooth map from $K_\epsilon(x,r)$ to $H(x,r)$. In addition, if $u \in G$, then $v_0(u \cdot a \cdot u^{-1}) \equiv u \cdot v_0(a) \cdot u^{-1}$.*

Proof of Lemma 4.3. The existence of a unique critical point of $\mathfrak{s}_a(\cdot)$ in $H(x,r)$ which obeys assertions (1) and (2) is a straightforward application of the contraction mapping principle, since one can consider the variational equation for $\mathfrak{s}_a(\cdot)$ as the following fixed point equation for v_0:

$$v_0 = -(P_- d_\Gamma^* P_- d_\Gamma + d_\Gamma d_\Gamma^*)_0^{-1} \nabla \mathfrak{s}_a|_{v=0} - R(a, v_0). \tag{4.5}$$

Here $(P_- d_\Gamma^* P_- d_\Gamma + d_\Gamma d_\Gamma^*)_0^{-1}$ is the Dirichlet inverse. (See, e.g., the proof of Proposition 8.2 of [22].)

For the proof of the third assertion, suppose that

$$\|P_- F_{\Gamma+a+v_0}\|^2_{L^2;B(x,r/2)} \le \tfrac{1}{2}\|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r/2)}.$$

Expand $F_{\Gamma+a+v_0} = F_{\Gamma+a} + d_{\Gamma+a}v_0 + v_0 \wedge v_0$. Thus,

$$\langle P_- F_{\Gamma+a}, P_- d_{\Gamma+a} v_0\rangle_{L^2;B(x,r/2)} \ge \tfrac{1}{8}\cdot\|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r/2)} - z\cdot\|v_0\|^4_{L^4;B(x,r/2)},$$

which implies that

$$\|P_- d_{\Gamma+a} v_0\|^2_{L^2;B(x,r/2)} \ge \tfrac{1}{8}\cdot\|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r/2)} - z\cdot\|v_0\|^4_{L^4;B(x,r/2)}.$$

This last equation implies that

$$\|P_- d_{\Gamma+a} v_0\|^2_{L^2;B(x,r)} \ge \tfrac{1}{8}\cdot\|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r/2)} - z\cdot\|v_0\|^4_{L^4;B(x,r/2)}.$$

Meanwhile, the variational equations which v_0 satisfy imply that

$$-\langle P_- F_{\Gamma+a}, P_- d_{\Gamma+a} v_0\rangle_{L^2;B(x,r)} \ge \|P_- d_{\Gamma+a} v_0\|^2_{L^2;B(x,r)} - z\cdot\|v_0\|^4_{L^4;B(x,r)}.$$

These last two equations imply that

$$\begin{aligned}\|P_- F_{\Gamma+a+v_0}\|^2_{L^2;B(x,r)} \le{}& \|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r)}\\ &- \tfrac{1}{16}\cdot\|P_- F_{\Gamma+a}\|^2_{L^2;B(x,r/2)} + z\cdot\|v_0\|^4_{L^4;B(x,r)}.\end{aligned}$$

Finally, the preceding equation with assertion (2) yields assertion (3).

The a priori estimates in assertion (4) are standard (see [17, Chapter 6] and the discussion in §9 of [22]). As for assertion (6), the continuity of the assignment of $v_0(a)$ to a is a consequence of the inverse function theorem applied to (4.5). (The argument here mimics the proof of Lemma 8.6 of [22].)

For the proof of assertion (5), look at the variational equations which are satisfied by $a' \equiv a + v(a)$. Using the Bianchi identity, these can be put in the following form:

$$d^*_{\Gamma+a'}(P_- F_{\Gamma+a'}) + d_\Gamma d_\Gamma^* a' = 0.$$

Differentiate this equation to obtain the second order equation

$$P_- d_\Gamma(d^*_{\Gamma+a'}(P_- F_{\Gamma+a'})) = 0.$$

By commuting derivatives, rewrite the preceding equation as

$$\nabla_\Gamma^*\nabla_\Gamma(P_- F_{\Gamma+a'}) + \mathfrak{R}\cdot P_- F_{\Gamma+a'} + P_- d_\Gamma^*([a', P_- F_{\Gamma+a'}]) = 0. \tag{4.6}$$

In (4.6), $\mathfrak{R}\cdot$ denotes a particular linear combination of the components of the Riemann curvature of the metric on TM; here, this curvature acts as an endomorphism of $P_- \bigwedge^2 T^*M$ (see Appendix C of [13]).

Using assertion (4) of Lemma 4.3, the estimate in question follows from (4.6) with the standard techniques in [17, Chapter 6].

To apply Lemma 4.3, recall that Uhlenbeck [23] gives $\kappa_0 > 0$ with the following significance: Let $P \to M$ be a principal G-bundle and let $[A] \in \mathfrak{B}(P)$ be given. For $\epsilon \in (0, \kappa_0)$, suppose that $x \in M$ and $r > 0$ are such that

$$\mathfrak{a}(A; x, r) \equiv \int_{B(x,r)} |F_A|^2 \cdot d\,\mathrm{vol} < \epsilon. \tag{4.7}$$

There exists a trivialization $\varphi\colon B(x,r) \times G \to P|_{B(x,r)}$ such that the $\mathfrak{g}$-valued 1-form $a(A) \equiv \varphi^* A - \Gamma$ obeys

$$\begin{aligned} &(1) \quad d_\Gamma^* a = 0, \\ &(2) \quad i^*_{\partial B(x,r)}(*a) = 0, \\ &(3) \quad \|\nabla_\Gamma a\|^2_{L^2;B(x,r)} + \|a\|^2_{L^2;B(x,r)} \le z_0 \cdot \|F_A\|^2_{L^2;B(x,r)}. \end{aligned} \tag{4.8}$$

Here z_0 is independent of A, x, and r.

The arguments in the proof of Lemma A.1 in [20] can be used to prove that the trivialization φ is unique up to $\varphi \to \varphi \circ u$, where $u \in G$ acts on $B(x,r) \times G$ as a constant gauge transformation. This last fact implies the following lemma.

Lemma 4.4. *Let M be a compact, oriented Riemannian 4-manifold. There exist $r_0 > 0$ and $\epsilon_0 > 0$, and for each integer $m \ge 0$, there exists $\xi_m < \infty$, and these have the following significance: Let $x \in M$, and let $r \in (0, r_0)$ and $\epsilon \in (0, \epsilon_0)$. Let $P \to M$ be a principal G-bundle, and let $\mathfrak{B}_\epsilon(P; x, r) \equiv \{[A] \in \mathfrak{B}(P)\colon (4.7) \text{ holds}\}$. There exists a smooth, $\mathfrak{a}$-decreasing homotopy $\phi_{(x,r)} \equiv \phi\colon [0,1] \times \mathfrak{B}'(P) \to \mathfrak{B}'(P)$ which induces a homotopy $\phi\colon [0,1] \times \mathfrak{B}_\epsilon(P; x, r) \to \mathfrak{B}_\epsilon(P; x, r)$ with the following properties: First, if $[A] \notin \mathfrak{B}_\epsilon(P; x, r)$, then $\phi(t, [A, h]) \equiv [A, h]$ for all $t \in [0,1]$. Second, for each $(t, [A]) \in [0,1] \times \mathfrak{B}_\epsilon(P; x, r)$, $\phi(t, A) \equiv [A + t \cdot \beta(\mathfrak{a}(A; x, r)/\epsilon) \cdot v(A)]$, where $\mathfrak{a}(A; x, r)$ is given by (4.7), and where $\beta(\cdot) \in C^\infty((0,\infty), [0,1])$ is identically 1 on $[0, 1/2]$ and vanishes identically on $[1, \infty)$. Here $[A, v(A)]$ defines a smooth section of the vector bundle $\mathfrak{A}(P) \times_{\mathfrak{G}(P)} L^2_3(\Omega^1(\mathrm{Ad}\, P)) \to \mathfrak{B}_\epsilon(P; x, r)$ which obeys the following:*

(1) The support of $v(A)$ is in $B(x,r)$ and $\|v(A)\|^2_A \le \xi_0 \cdot \|\nabla \mathfrak{a}_A\|^2_{A;B(x,r)}$, where $\|\nabla \mathfrak{a}_A\|_{A;B(x,r)}$ is the norm of the restriction of $\nabla \mathfrak{a}_A$ to the subspace of $L^2_1(\Omega^1(\mathrm{Ad}\, P))$ with compact support in $B(x,r)$.

(2) If $\|P_- F_{\varphi(1,A)}\|^2_{L^2;B(x,r/2)} \le \frac{1}{2} \|P_- F_A\|^2_{L^2;B(x,r/2)}$ for $[A] \in \mathfrak{B}_{\epsilon/2}(P; x, r)$, then

$$\|P_- F_{\phi(1,A)}\|^2_{L^2;B(x,r)} \le (1 + z \cdot \epsilon^2) \cdot \|P_- F_A\|^2_{L^2;B(x,r)} - \tfrac{1}{16} \cdot \|P_- F_A\|^2_{L^2;B(x,r/2)}.$$

(3) *For* $[A]$ *as above, when* $y \in B(x, 3r/4)$, *then* $|\nabla^{(m)}_{\phi(1,A)} F_{\phi(1,A)}|(y) \leq \xi_m \cdot r^{-m-2} \cdot \|F_A\|_{L^2;B(x,r)}$.

(4) *For* $[A]$ *as above, when* $y \in B(x, 3r/4)$, *then*

$$|P_- F_{\phi(1,A)}|(y) + r \cdot |\nabla_{\phi(1,A)}(P_- F_{\phi(1,A)})|(y) \leq \xi_0 \cdot r^{-2} \cdot \|P_- F_{\phi(1,A)}\|_{L^2;B(x,r)}.$$

Lemma 4.4 is somewhat unsatisfactory vis-a-vis its application to the proof of Proposition 4.2. The condition for appeal to Lemma 4.4 is a condition on the L^2-norm, over the ball, of the total curvature. The proof of Proposition 4.2 requires a condition on a norm over the ball for only the anti-self-dual part of the curvature. This problem is circumvented with

Lemma 4.5. *There exists* $q > 0$ *with the following significance: Let* M *be a compact, Riemannian manifold with boundary. Let* $P \to M$ *be a principal* G*-bundle; let* $\delta > 0$ *be given, and let* $U \subset \mathfrak{B}(P)$ *be a compact set of orbits such that* $\|P_- F_A\|^2_{L^2} \geq \delta$ *for all* $[A] \in U$. *Fix* $\epsilon > 0$. *For any* $r > 0$ *sufficiently small, there exists a set of disjoint balls* $\{B(x[j], r) \subset \mathrm{int}(M)\}$ *with the following two properties: (1) For each* j, $U \subset \mathfrak{B}_\epsilon(P; x[j], r)$; *(2) For each* $[A] \in U$,

$$\sum_j \|P_- F_A\|^2_{L^2;B(x[j],r/2)} \geq q \cdot \|P_- F_A\|^2_{L^2}.$$

Proof of Lemma 4.5. For r much less than the injectivity radius of M, define a lattice $\Gamma(r) \subset M$ to be a set of points in $\mathrm{int}(M)$ with the following properties:

(1) For any pair of disjoint points $x, y \subset \Gamma(r)$, $\mathrm{dist}(x, y) > 2 \cdot r$.

(2) Every point in M has distance less than $4 \cdot r$ from a point in $\Gamma(r)$.

(3) $\mathrm{dist}(\partial M, \Gamma(r)) > 2 \cdot r$.

Construct a lattice $\Gamma(r)$ and let $\Gamma(r) \equiv \{x[j]\}$. Observe that the set of balls $\{B(x[j]; 4 \cdot r)\}$ forms an open cover of M.

Next, let $f \in L^2_1(M)$ and consider $u(x) \equiv \|f\|_{L^2, B(x,r)}$ as a function of $x \in M$. Observe that u is Lipschitz, and

$$|du|(x) \leq \|df\|_{L^2;B(x,r)}.$$

Thus, for each j,

$$\|f\|_{L^2;B(x[j],r/2)} \geq z_0 \cdot \|f\|_{L^2;B(x[j],4r)} - z_1 \cdot r \cdot \|df\|_{L^2;B(x[j],4r)},$$

with z_0, z_1 independent of f, x and r. Now, sum both sides of this last equation over the set of points in Γ. The result is the following inequality with constants $z_0 > 0$ and $z_1 < \infty$:

$$\sum_j \|f\|_{L^2;B(x[j],r/2)} \geq z_0 \cdot \|f\|_{L^2;M} - z_1 \cdot r \cdot \|df\|_{L^2;M}. \tag{4.9}$$

Consider the compact set $U \subset \mathfrak{B}$. The compactness of U insures the existence of $r_0 > 0$ with the property that $U \subset \mathfrak{B}_\epsilon(P; x, r)$ for any $x \in M$ and $r < r_0$. The compactness of U also insures the existence of $z_2 < \infty$ with the property that $\|d|F_A|\|_{L^2;M} < z_2$ for any $[A] \in U$. Lemma 4.5 follows from these last observations with (4.9).

Before beginning the proof of Proposition 4.2, it is necessary to digress and discuss a modification of the construction of the map θ_+ in (4.1). To begin, let $k_0 \geq 0$ be the minimal, nonnegative integer for which (k_0, η) are admissible as characteristic classes for a principal G-bundle $P \to M$. The modification to the map θ_+ involves replacing the connection $A(w)$ in (4.2) by the connection $A'(w)$ which differs from $A(w)$ in the ball $B(x, 4 \cdot \lambda/t)$ (recall that $\lambda = t \cdot s$). In this ball,

$$A'(w) \equiv A(w) + v(w). \tag{4.10}$$

Here, $v \equiv v(w) = \beta_{4\lambda/t} \cdot d^*_{A(w)} u$ with $u \in L^2_3(P_-\Omega^2(\operatorname{Ad} P(w)|_{B(x,4\lambda/t)}))$ obeys the following differential equation in $B(x, 5 \cdot \lambda/t)$:

$$\begin{aligned} P_- d_{A(w)} d^*_{A(w)} u + P_-(d^*_{A(w)} u \wedge d^*_{A(w)} u) \\ + P_- F_{A(w)} - (1 - \beta_\lambda) \cdot P_- F_{A_0} = 0, \end{aligned} \tag{4.11}$$

with $u \equiv 0$ on $\partial B(x, 5 \cdot \lambda/t)$. The cut-off function $\beta_\lambda(\cdot)$ is defined after (4.1).

With A_0 completely arbitrary, the section u will exist provided that λ and t are appropriate. For the applications below, the constraints on λ and t are summarized in Lemma 4.7, below. The existence of u is established by mimicking the proof of Lemma 5.2, below. Indeed, for an L^2_2-connection A on a principal G-bundle over M, and for $x \in M$ and $\rho > 0$, define the range Banach space as the completion of

$$\begin{aligned} W(x, \rho) \equiv \{\omega \in L^2(P_-\Omega^2(\operatorname{Ad} P)|_{B(x,\rho)}): \\ \sup_{y \in B(x,\rho)} \|\operatorname{dist}(\cdot, y)^{-2}\omega\|_{L^1;B(x,\rho)} < \infty\} \end{aligned}$$

with metric

$$|\omega|_W \equiv \|\omega\|_{L^1;B(x,\rho)} + \sup_{y \in B(x,\rho)} \|\operatorname{dist}(\cdot, y)^{-2}\omega\|_{L^1;B(x,\rho)}. \tag{4.12}$$

Define the domain Banach space as the completion of

$$U(A; x, \rho) \equiv \{u \in L^2_2 \cap L^\infty(P_-\Omega^2(\operatorname{Ad} P)|_{B(x,\rho)}): u \equiv 0 \text{ an } \partial B(x, \rho)\}$$

with the norm

(4.13)

$$|u|_U \equiv \|\nabla_A u\|_{L^2;B(x,\rho)} + \sup_{y \in B(x,\rho)} \|\operatorname{dist}(y, \cdot)^{-1}\nabla_A u\|_{L^2;B(x,\rho)} + \sup_{B(x,\rho)} |u|(\cdot).$$

Given $\omega \in W(x, \rho)$, the solvability of the equation

$$P_- d_A(d_A^* u) + P_-(d_A^* u \wedge d_A^* u) + \omega = 0 \tag{4.14}$$

for $u \in U(A; x, \rho)$ is established by copying the contraction mapping argument in the proof of Lemma 5.2. The result is summarized in the lemma below; for the proof, the modification of the proof of Lemma 5.2 is left to the reader.

Lemma 4.6. *Let M be a compact, oriented, Riemannian 4-manifold. There exists $\rho_0, \epsilon_0 > 0$ and $z_0 < \infty$ with the following significance: Let G be a compact Lie group, and let $P \to M$ be a principal G-bundle and for $x \in M$ and $\rho \in (0, \rho_0)$, let $\mathfrak{A}(W(x, \rho)) \equiv \{A \in \mathfrak{A}(P)\colon |P_- F_A|_{W(x,\rho)} < \epsilon_0\}$ let $W_{\epsilon_0}(x, \rho) \equiv \{\omega \in W(x, \rho)\colon |\omega|_{W(x,\rho)} < \epsilon_0\}$. For $A \in \mathfrak{A}(W(x, \rho))$ and for $\omega \in W_{\epsilon_0}(x, \rho)$, (4.14) has a unique solution $u \in U(A; x, \rho) \cap L_3^2(P_-\Omega^2(\mathrm{Ad}\, P|_{B(x,4\rho/s)}))$ satisfying $|u|_U < z_0 \cdot |\omega|_{W(x,\rho)}$. Furthermore, the assignment of $(A, \omega) \in \mathfrak{A}(W(x, \rho)) \times W_{\epsilon_0}(x, \rho)$ to $u \in L_3^2(P_-\Omega^2(\mathrm{Ad}\, P|_{B(x,4\rho/s)}))$ defines a smooth, $\mathfrak{G}(P)$-equivariant map.*

The application of Lemma 4.6 is to (4.11) where $A \equiv A(w)$ and where $\rho \equiv 4 \cdot \lambda/t$. In this case, additional conclusions concerning u are derivable. To state these conclusions, it is necessary to construct from the data $w \equiv [[A_0, h_0], f_0, \varphi, s([A_0, h_0]), t([A_0, h_0]), [A_1, h_1]]$ an isomorphism

$$X(\varphi, f_0, [A_0, h_0], h_1)\colon P_+\Omega^2(\mathrm{Ad}\, P_+|_\theta) \to P_-\Omega^2(\mathrm{Ad}\, P_0|_x).$$

To define the isomorphism, observe first that an isomorphism, $i_1\colon \mathrm{Ad}\, P_+|_s \to \mathrm{Ad}\, P_0|_x$ is obtained as follows: Send $\sigma \in \mathrm{Ad}\, P_+|_s$ to $h_0 \cdot h_1^{-1} \cdot \sigma \cdot h_1 \cdot h_0^{-1} \in \mathrm{Ad}\, P_0|_{x_0}$, and then parallel transport the result using the connection A_0 along the path φ to $\mathrm{Ad}\, P_0|_x$. Next, fix the inverse stereographic projection map $\Phi\colon \mathbf{R}^4 \to S^4 \backslash s$. Let $\omega \in P_+ \bigwedge^2 T^*S^4|_s$; by parallel transport using a Euclidean metric on $S^4 \backslash n$, ω defines a section of $P_+ \bigwedge^2 T^*S^4|_{S^4\backslash n}$, and $\Phi^*\omega$ defines a section of $P_+ \bigwedge^2 T^*(\mathbf{R}^4 \backslash 0)$. Note that

$$\omega' \equiv \lim_{x\to 0} |x|^4 \cdot P_-(d|x| \wedge *(d|x| \wedge \omega))$$

is well defined. By parallel transport using the Levi-Civita connection along φ the given fiducial frame $f_0 \in \mathrm{Fr}\, M|_{x_0}$ defines a frame $f \in \mathrm{Fr}\, M|_x$, and thus a Gaussian coordinate system y_i from a ball centered at x to $\mathbf{R}^4$ which takes x to 0. Thus, an isomorphism $i_2\colon P_+ \bigwedge^2 T^*S^4|_s \to P_- \bigwedge^2 T^*S^4|_x$ is defined by sending ω to $y_i^*\omega'$. Finally, set $X(\varphi, f_0, [A_0, h_0], h_1) \equiv i_1 \otimes i_2$.

Lemma 4.7. *Let G be a compact Lie group, and let M be a compact, Riemannian manifold. There exist $c_2(G)$, constants $r_0(M, G) \in (0, 1/8)$ and $z_0(M, G) < \infty$, and, given $\xi, \xi' < \infty$, there exists $\alpha_0 \in (0, 1/8)$ which are such that the following is true: Let $x \in M$, and let $r < r_0$. Let $P \to M$ be a principal G-bundle, and let $[A_0] \in \mathfrak{B}(P; x, r)$ and all $y \in B(x, 3r/4)$ obey*

$$|F_{A_0}|(y) + r \cdot |\nabla_{A_0} F_{A_0}|(y) < \xi \cdot r^{-2}, \quad |P_- F_{A_0}|(y) + r \cdot |\nabla_{A_0} P_- F_{A_0}|(y) < \xi' \cdot r^{-2}.$$

Let $\lambda > 0$ *and* $t > 0$ *be given such that* $\lambda/r \equiv \alpha \in (0, \alpha_0)$ *and* $t \in (\alpha, \sqrt{\alpha})$. *Let* $\varphi\colon [0,1] \to M$ *be a smooth path from* x_0 *to* x, *let* $f_0 \in \operatorname{Fr} M|_{x_0}$, *and let* $h_0 \in P|_{x_0}$. *Set* $w \equiv [[A_0, h_0], f_0, \varphi, s \equiv \lambda/t, t, [A_1, h_1]]$. *Use* (4.2) *to construct the connection* $A(w)$. *Then, the connection* $A(w) \in \mathfrak{A}(W(x, 4\cdot\lambda/t))$ *and* $P_-F_{A(w)} - (1-\beta_\lambda)\cdot P_-F_{A_0} \in W_{\rho_0}(x,\rho)$. *Construct* $u \equiv u(w) \in L^2_3(P_-\Omega^2(\operatorname{Ad} P(w)|_{B(k,4\lambda/t)}))$ *as given by Lemma* 4.6 *and* (4.11), *and then construct the connection* $A'(w,t)$ *as in* (4.10). *Then*

$$\begin{aligned}
&\big|\|P_-F_{A'(w)}\|^2_{L^2} - \|P_-F_{A_0}\|^2_{L^2} \\
&\qquad - c(G)\cdot\lambda^2\cdot(P_-F_{A_0}(x), X(\varphi, f_0, [A_0,h_0], h_1)\cdot P_+F_{A_1}(s))\big| \\
&\qquad \le z_0\cdot(t^4 + \xi^2(1+\xi^2)\cdot\alpha^4 + \xi'\cdot(1+\xi)^4\cdot\alpha^3/t).
\end{aligned}$$

Proof of Lemma 4.7. The assertion that the constants α_0, R_0, and r_0 of the lemma can be chosen so that $(A(w), P_-F_{A(w)} - (1-\beta_\lambda)\cdot P_-F_{A_0}) \in \mathfrak{A}(W(x,4\cdot\lambda/t)) \times W_{\rho_0}(x,\rho)$ follows from an algebraic calculation using the explicit formula in (4.4) of [21] for the $\mathfrak{g}$-valued 1-form $a_1 \equiv \lambda^*\phi(A_1,h_1)^*A_1$ on $\mathbf{R}^4\backslash 0$. Use said formula with the expression below for $P_-F_{A(w)}$ on $B(x;r)$.

$$\begin{aligned}
&P_-F_{A(w)} - (1-\beta_\lambda)\cdot P_-F_{A_0} \\
&\quad = P_-(d\beta_{\lambda/t}\wedge a_1 - \beta_{\lambda/t}\cdot(1-\beta_{\lambda/t})\cdot a_1\wedge a_1 - d\beta_\lambda\wedge a_0 \\
&\qquad - \beta_\lambda\cdot(1-\beta_\lambda)\cdot a_0\wedge a_0 + (1-\beta_\lambda) \\
&\qquad \cdot\beta_{\lambda/t}\cdot(a_0\wedge a_1 + a_1\wedge a_0)) \\
&\quad + \beta_{\lambda/t}\cdot P_-F_{\Gamma+a_1}.
\end{aligned} \tag{4.15}$$

By a straightforward calculation, one obtains the following bounds:

$$\begin{aligned}
&|P_-F_{A(w)}|_{W(x,4\lambda/t)} \\
&\quad \le z_0\cdot(\alpha^2/t^2\cdot\xi' + \alpha^2\cdot\xi + \alpha^4\cdot\xi^2 + t^2 + t^4 \\
&\qquad + |\ln t|^{1/2}\cdot(\alpha^2\cdot\xi + \lambda^2)), \\
&|P_-F_{A(w)} - (1-\beta_\lambda)\cdot P_-F_{A_0}|_{W(x,4\lambda/t)} \\
&\quad \le z_0\cdot(\alpha^2\cdot\xi + \alpha^4\cdot\xi^2 + t^2 + t^4 + |\ln t|^{1/2}\cdot(\alpha^2\cdot\xi + \lambda^2)).
\end{aligned} \tag{4.16}$$

Here, $\alpha \equiv \lambda/r$. (4.16) gives the first assertion of the lemma.

It is important for future remarks to observe that given $\epsilon > 0$, the constants α_0, R_0 and r_0 of Lemma 4.7 can be chosen so that the left-hand side of (4.16) obeys

$$|P_-F_{A(w)}|_{W(x,4\lambda/t)} + |(1-\beta_\lambda)\cdot P_-F_{A_0}|_{W(x,4\lambda/t)} < \epsilon. \tag{4.17}$$

Consequently, u, as given by Lemma 4.6, obeys

$$|u(w)|_{U(A(w);x,4\lambda/t)} < z_0\cdot\epsilon. \tag{4.18}$$

The number ϵ will be restricted by requiring, first of all, that

$$\|\nabla_{A(w)}\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2 \leq z_0 \cdot \|d^*_{A(w)}\omega\|_{L^2}^2 \tag{4.19}$$

whenever $\omega \in L_1^2(P_-\Omega^2(\operatorname{Ad} P(w)))$ obeys $\omega \equiv 0$ on $\partial B(x, r)$.

To complete the proof of Lemma 4.7, it is necessary to generate the asserted a priori estimates for u; since

$$\begin{aligned} &\|P_-F_{A'(w)}\|_{L^2}^2 - \|(1-\beta_\lambda)\cdot P_-F_{A_0}\|_{L^2}^2 \\ &= 2\cdot\langle -P_-F_{A_0}, d\beta_{4\lambda/t}\wedge d_A^*u + \beta_{4\lambda/t}\cdot(1-\beta_{4\lambda/t})\cdot d_A^*u\wedge d_A^*u\rangle_{L^2} \\ &\quad + \| - d\beta_{4\lambda/t}\wedge d_A^*u + \beta_{4\lambda/t}\cdot(1-\beta_{4\lambda/t})\cdot d_A^*u\wedge d_A^*u\|_{L^2}^2, \end{aligned} \tag{4.20}$$

the estimates start with

Lemma 4.8. *The constants* α_0, R_0 *and* r_0, z_0 *can be chosen as in Lemma* 4.7 *so that under the assumptions of Lemma* 4.7,

$$\|d_A^*u\|_{L^2} + \|u\|_{L^2} \leq z_0\cdot(\lambda\cdot t\cdot(1+t^2) + \xi\cdot\alpha^2\cdot\lambda + \xi^2\cdot\alpha^4\cdot\lambda + \xi\cdot\alpha^2\cdot\lambda/t + \lambda^3).$$

Proof of Lemma 4.8. Contract both sides of (4.14) with u, then integrate over $B(x, 4\cdot\lambda/t)$ and then integrate by parts. Lemma 4.6 and (4.16) insure that the constants in Lemma 4.7 can be chosen so that the first term on the left-hand side of the resulting equation dominates twice the second term. Then, Holder's inequality gives

$$\|d_A^*u\|_{L^2}^2 \leq z\cdot\|u\|_{L^4}\cdot\|P_-F_A\|_{L^{4/3}}.$$

This last equation, plus (4.19) and (3.4) and some simple algebra, yields the lemma.

With the estimate from the preceding lemma, one immediately bounds

$$|\langle P_-F_{A_0}, \beta_{4\lambda/t}\cdot(1-\beta_{4\lambda/t})\cdot d_A^*u\wedge d_A^*u\rangle_{L^2}| + \|d\beta_{4\lambda/t}\wedge d_A^*u\|_{L^2}^2$$

by

$$\begin{aligned} &z_0\cdot(t^4 + t^8 + \xi^2\cdot\alpha^4\cdot t^2 + \xi^4\cdot\alpha^8\cdot t^2 + \xi^2\cdot\alpha^4 + r^4\cdot\alpha^4\cdot t^2) \\ &\quad + z_0\cdot\xi'\cdot(\alpha^2\cdot t^2 + \xi^2\cdot\alpha^6 + \xi^4\cdot\alpha^{10} + \xi^2\cdot\alpha^6/t^2 + \alpha^6\cdot r^4). \end{aligned} \tag{4.21}$$

To bound the last term in (4.20), introduce a smooth cut-off function $\theta(\cdot) \in C^\infty([0,\infty),[0,1])$ which has support in $[1/2, 1]$ and which is identically 1 on the support of $d\beta$. Let $\theta_{4\lambda/t}$ be the function on $B(x, 4\cdot\lambda/t)$ which sends y to $\theta(\operatorname{dist}(y,x)/(4\cdot\lambda/t))$.

Consider that (4.14) and (4.11) imply the following equation for $u' \equiv \theta_{4\lambda/t}\cdot u$:

$$\begin{aligned} &P_-d_{A_0}(d_{A_0}^*u') + P_-(d_{A_0}^*u'\wedge d_{A_0}^*u) \\ &\quad = P_-d_{A_0}^*(d\theta_{4\lambda/t}\wedge u) + P_-(d\theta_{4\lambda/t}\wedge d_{A_0}^*u) \\ &\qquad + P_-(*(d\theta_{4\lambda/t}\wedge u)\wedge d_{A_0}^*u). \end{aligned} \tag{4.22}$$

Using Lemma 4.6 and (4.16), one finds that the constants α_0, R_0, z_0 and r_0 of Lemma 4.7 can be chosen so that (4.22) gives an a priori bound on $\|d_A^* u'\|_{L^4}^4$ by the expression in (4.21). The arguments are practically the same as those in the proof of Lemma 3.7 of [20] and they are left to the reader (use (3.4)).

To complete the proof of Lemma 4.7, use integration by parts and Lemma 4.8 to obtain

$$\begin{aligned}&|-\langle P_- F_{A_0}, d\beta_{4\lambda/t} \wedge d_A^* u\rangle_{L^2} + \langle P_- F_{A_0}, \beta_{4\lambda/t} \cdot P_- d_A d_A^* u\rangle_{L^2}| \\ &\quad \leq z_0 \cdot \xi' \cdot \left(\alpha^3 \cdot \frac{1+t^2}{t} + \xi \cdot \frac{\alpha^5}{t^2} + \xi^2 \cdot \frac{\alpha^7}{t^2} + \xi \cdot \frac{\alpha^5}{t^3} + r \cdot \frac{\alpha^5}{t^3}\right).\end{aligned} \tag{4.23}$$

With (4.11) for u, (4.23) and Lemma 4.8 imply that

$$\begin{aligned}&|\langle -P_- F_{A_0}, d\beta_{4\lambda/t} \wedge d_A^* u\rangle_{L^2} - \langle P_- F_{A_0}, P_- F_{A(w)} - (1-\beta_\lambda) \cdot P_- F_{A_0}\rangle_{L^2}| \\ &\quad \leq z_0 \cdot \xi' \cdot \left(\alpha^3 \cdot \frac{1+t^2}{t} + \xi \cdot \frac{\alpha^5}{t^2} + \xi^2 \cdot \frac{\alpha^7}{t^2} + \xi \cdot \frac{\alpha^5}{t^3}\right. \\ &\qquad \left. + r \cdot \frac{\alpha^5}{t^3} + \alpha^2 \cdot t^2 + \xi^2 \cdot \alpha^6 + \xi^4 \cdot \alpha^{10} + \xi^2 \cdot \frac{\alpha^6}{t^2} + r^4 \cdot \alpha^6\right).\end{aligned} \tag{4.24}$$

Equation (4.15) is used to further evaluate (4.24); the contribution to

$$\langle P_- F_{A_0}, P_- F_{A(w)} - (1-\beta_\lambda) \cdot P_- F_{A_0}\rangle_{L^2}$$

from all but the first term on the right-hand side of (4.15) is bounded in absolute value by

$$z_0 \cdot \xi' \cdot (\alpha^2 \cdot t^2 + \xi \cdot \alpha^4 + \xi^2 \cdot \alpha^6 + \xi \cdot \alpha^4/t^2 + r^4 \cdot \alpha^6/t^2). \tag{4.25}$$

Finally, the contribution from the first term on the right-hand side of (4.24) can be estimated by using Taylor's theorem with remainder to expand $P_- F_{A_0}$ about the point x:

$$\begin{aligned}&|\langle P_- F_{A_0}, P_-(d\beta_{\lambda/t} \wedge a_1)\rangle_{L^2} \\ &\qquad + c(G) \cdot (P_- F_{A_0}(x), X(\varphi, f_0, [A_0, h_0], h_1) \cdot P_- F_{A_1}(s))| \\ &\quad \leq z_0 \cdot \xi' \cdot \alpha^3/t.\end{aligned} \tag{4.26}$$

(4.21) and (4.23)–(4.26) complete the proof of Lemma 4.7.

Proof of Proposition 4.2. The first assertion of the proposition is a consequence of Lemma 4.7. To prove the second assertion of Proposition 4.2, consider a principal G-bundle $P \to M$ and a compact set $U \subset \mathfrak{B}'(P)$ which obeys $\sup_{[A,h]\in U} \|P_- F_A\|_{L^2}^2 = \delta > 0$. For $\delta' < \delta$, let $U(\delta') \equiv \{[A,h] \in U : \|P_- F_A\|_{L^2}^2 \geq \delta'\}$. The proof of the second assertion of Proposition 4.2 begins with

Lemma 4.9. *Let $\epsilon_0 > 0$ be as in Lemma* 4.4. *There exist constants $\epsilon_1 \in (0, \epsilon_0)$ and $z_1 > 0$ with the following significance: Let $P \to M$ be a principal G-*

bundle and let $U \subset \mathfrak{B}'(P)$ be a compact set which obeys $\sup_{[A,h]\in U} \|P_-F_A\|^2_{L^2} = \delta > 0$. *For r sufficiently small, and less than r_0 of Lemma 4.7, and for $\epsilon \le \epsilon_1$, invoke Lemma 4.5 for the compact set $U(\delta/4)$. Since the balls $\{B(x[j],r)\}$ are disjoint, apply, for each j, the homotopy $\phi_{(x(j),r)}$ of Lemma 4.4 to the compact set U to construct a new compact set, $U_1 \subset \mathfrak{B}'(P)$, which is homotopic in $\mathfrak{B}'(P)$ to U. This multiple application of Lemma 4.4 gives a homotopy $\Psi\colon [0,1]\times\mathfrak{B}'(P) \to \mathfrak{B}'(P)$ and $U_1 = \Psi(1,U)$. The set U_1 has the following properties:*

(1) $U_1\backslash U_1(\delta/4) \equiv U\backslash U(\delta/4)$.

(2) $\mathfrak{a}(\Psi(t,[A,h])) \le \mathfrak{a}([A,h])$ *for all* $(t,[A,h]) \in [0,1]\times U$.

(3) *For all $(t,[A,h]) \in [0,1]\times U$, the restriction to $M\backslash\bigcup_j B(x[j],r)$ of $\Psi(t,[A,h])$ equals that of $[A,h]$.*

(4) *The connections whose orbits lie in $U_1(\delta/2)$ obey Lemma 4.4's a priori estimates on each $B(x[j],3/4\cdot r)$.*

(5) *Let q be as in Lemma 4.5. For $[A,h]\in U_1(\delta/2)$, either*

$$\sum_j \|P_-F_A\|^2_{L^2;B(x[j],r/2)} \ge \tfrac{1}{4}\cdot q\cdot\|P_-F_A\|^2_{L^2},$$

or else $\|P_-F_A\|^2_{L^2} < (1-z_1)\cdot\delta$.

Proof of Lemma 4.9. The first four assertions only summarize assertions of Lemmas 4.4 and 4.5. As for the fifth assertion, suppose that $[A',h'] \in U(\delta/2)$ and $[A,h] \equiv \Psi(1,[A',h'])$. Let Υ denote the set of centers $\{x[j]\}$ and let Υ' denote the subset of $\{x[j]\}$ for which

$$\|P_-F_A\|^2_{L^2;B(x[j],r/2)} \ge \tfrac{1}{2}\|P_-F_{A'}\|^2_{L^2;B(x[j],r/2)}.$$

If $\sum_j \|P_-F_A\|^2_{L^2;B(x[j],r/2)} < \frac{1}{4}\cdot q\cdot\|P_-F_A\|^2_{L^2}$, then

$$\sum_{\Upsilon'} \|P_-F_{A'}\|^2_{L^2;B(x[j],r/2)} < \tfrac{1}{2}\cdot q\cdot\|P_-F_A\|^2_{L^2} \le \tfrac{1}{2}\cdot q\cdot\|P_-F_{A'}\|^2_{L^2}.$$

Consequently,

$$\sum_{\Upsilon\backslash\Upsilon'} \|P_-F_{A'}\|^2_{L^2;B(x[j],r/2)} > \tfrac{1}{2}\cdot q\cdot\|P_-F_{A'}\|^2_{L^2}.$$

Then, assertion (3) of Lemma 4.4 asserts that

$$\sum_{\Upsilon\backslash\Upsilon'} \|P_-F_A\|^2_{L^2;B(x[j],r)} \le (1+\epsilon^2)\cdot\sum_{\Upsilon\backslash\Upsilon'} \|P_-F_{A'}\|^2_{L^2;B(x[j],r)} - z\cdot\|P_-F_{A'}\|^2_{L^2},$$

where $z \equiv q/2048$. Due to assertion (3) of Lemma 4.9, this last equation means that

$$\|P_-F_A\|^2_{L^2} \le (1+\epsilon^2 - z)\cdot\|P_-F_{A'}\|^2_{L^2} \le (1-z_1)\cdot\delta,$$

as claimed.

Choose a base point $x_0 \in M$ which is disjoint from each $B(x[j], r)$. For each j, choose a smooth path $\varphi'[j]$ from x_0 to $x[j]$ which does not intersect $B(x[j'], r)$ unless $j' = j$. If $x \in B(x[j], r)$, specify a path $\varphi[x]$ from x_0 to x by first traveling from x_0 to $x[j]$ along $\varphi'[j]$, and then by traveling from $x[j]$ to x along the unique short, radial geodesic between them.

Use the chosen path to $x[j]$, and the fiducial frame in $TM|_{x_0}$ to define an orthonormal frame $f_j \in P_- \bigwedge^2 T^*|_{x[j]}$, and, by parallel transport with the Levi-Civita connection along the radial geodesics from $x[j]$, trivialize $P_- \bigwedge^2 T^*|_{B(x[j],r)}$.

A subdivision of each of the balls $B(x[j], r)$ is required. Let N denote the number of balls $\{B(x[j], r)\}$. Fix $r_1 \in (0, r)$ to be further specified shortly. Fix j and let

$$U_1(\delta/2; j) \equiv \{[A, h] \in U_1(\delta/2) \colon \|P_- F_A\|^2_{L^2; B(x[j], 3r/4)} \geq 1/(32 \cdot N) \cdot q \cdot \delta\}.$$

Due to Lemma 4.9, for fixed $[A, h] \in U_1(\delta/2)$, either

$$\sum_{j \colon [A,h] \in U_1(\delta/2; j)} \|P_- F_A\|^2_{L^2; B(x[j], r/2)} \geq \tfrac{1}{8} \cdot q \cdot \|P_- F_A\|^2_{L^2}, \tag{4.27}$$

or else $\|P_- F_A\|^2_{L^2} < (1 - z_1) \cdot \delta$. Using Lemma 4.5, and the compact set $U_1(\delta/4, j) \subset \mathfrak{B}$, construct a set of disjoint balls $\{B(x[j,k], r_1)\} \subset B(x[j], 3r/4)$ with the property that for any $[A, h] \in U_1(\delta/2, j)$,

$$\sum_k \|P_- F_A\|^2_{L^2; B(x[j,k], r_1)} > q \cdot \|P_- F_A\|^2_{L^2; B(x[j], 3r/4)}. \tag{4.28}$$

A further subdivision of $B(x[j,k], r_1)$ into $m(G)$ smaller balls $\{B[j,k,\beta] \equiv B(x[j,k,\beta], r_1/m)\}_{\beta=1,\cdots,m}$ is required. The number $m(G)$ is provided by

Lemma 4.10. *Let G be a simple, compact Lie group. There exists $c_1(G) > 0$, an integer $m < \infty$ and m homomorphisms $\{\rho[\beta] \colon \mathrm{SU}(2) \to G\}$ which have the following properties: First, each homomorphism $\rho[\beta]$ generates $\pi_3(G)$, so any pair are conjugate in G. Second, given $Y \in \mathfrak{g} \otimes \mathfrak{su}(2)$ and an orthonormal basis $\{\sigma_i\}_{i=1}^3$ for $\mathfrak{su}(2)$, there exists β such that $(\sum_{i=1}^3 \rho[\beta]_* \sigma_i \otimes \sigma_i, Y)_{\mathfrak{g} \otimes \mathfrak{su}(2)} \leq -c_1(G) \cdot |Y|$. Here, $\rho[\beta]_* \colon \mathfrak{su}(2) \to \mathfrak{g}$ is the induced Lie algebra homomorphism, and $(\cdot, \cdot)_{\mathfrak{g} \otimes \mathfrak{su}(2)}$ is the natural inner product.*

Proof of Lemma 4.10. Using the fact that G is compact, this follows by a slight modification of the proof of Proposition 6.2 in [20]; the details are left to the reader.

For $\beta = 1, \cdots, m$, fix $h_\beta \in G$ so that $\rho[\beta]$ is conjugate by h_β in G to $\rho[1]$. Fix $[A_1, h_1] \in \mathfrak{B}'(P_{G+})$ as described in the proof of Proposition 4.1, and let $[A_1, h_1]_\beta \equiv [A_1, h_1 \cdot h_\beta]$. Focus attention on one ball, $B(x[j,k,\beta], r_1/m)$, and choose an isometry to identify the 3-dimensional vector space $P_+ \bigwedge^2 T^*S^4|_s$ with $\mathfrak{su}(2)$. This can be done so that

$$h_\beta \cdot h_1^{-1} \cdot P_+F_{A_1}(s) \cdot h_1 \cdot h_\beta^{-1} = \sum_{i=1}^{3} \rho[\beta]_* \sigma_i \otimes \sigma_i, \tag{4.29}$$

where $\{\sigma_i\}$ is a basis for $\mathfrak{su}(2)$.

Let ξ_0 be as in Lemma 4.4, and choose ϵ_1 so that $\epsilon_1 \cdot \xi_0 < 1$. Let ξ, ξ' both equal 1 in Lemma 4.7, and introduce the constant α_0 of that lemma. Suppose that $R > 1$ is given, and suppose that there is defined a continuous map $\alpha \equiv \alpha[j, k, \beta] \colon U_1 \to (0, \alpha_0)$.

Let $[A, h] \in U_1$, and set

$$w \equiv [[A, h], f_0, \varphi[x(j, k, \beta)], s \equiv r_1/(R \cdot m), t \equiv R \cdot \alpha, [A_1, h_1]_\beta].$$

Next, construct the connection $A'(w)$ of (4.10). According to Lemma 4.7,

$$\begin{aligned} \|P_-F_{A'(w)}\|_{L^2}^2 \le \|P_-F_A\|_{L^2}^2 &+ c(G) \\ &\cdot \alpha^2 \cdot (r_1^2/m^2 \cdot P_-F_A(x[j, k, \beta]), \\ &X(f, h_0, h_1) \cdot h_1 \cdot h_\beta \cdot h_1^{-1} \cdot P_+F_{A_1}(s) \cdot h_1 \cdot h_\beta^{-1} \cdot h_1^{-1}) \\ &+ z_0 \cdot (R^4 \cdot \alpha^4 + \alpha^4 + \xi'[j, k, \beta] \cdot \alpha^2/R), \end{aligned} \tag{4.30}$$

where $\xi'[j, k, \beta] \equiv r_1^2/m^2 \cdot \sup_{B(x[j,k,\beta], r_1/m)} |P_-F_A|$.

To evaluate (4.30), introduce the number $\xi^* \equiv \|P_-F_A\|_{L^2; B(x[j], 3r/4)}$, and observe that

$$z_1 \cdot r^2 \cdot \sup_{B(x[j], 3r/4)} |P_-F_A| \ge \xi^* \ge z_2 \cdot r^2 \cdot \sup_{B(x[j], 3r/4)} |P_-F_A|, \tag{4.31}$$

with constants z_1, z_2 being independent of r, $x[j]$, the connection A and the principal G-bundle P. Here, the second inequality is a consequence of Lemma 4.4, and the first inequality is automatic. Due to (4.28) and (4.31) there exists a constant $z_4 > 0$ which is independent of r, $x[j]$, $x[j, k]$, r_1, the connection A and the principal G-bundle P and which is such that when $[A, h] \in U_1(\delta/4, j)$, then at least z_4 of the balls $\{B(x[j, k], r_1)\}$ must have

$$\sup_{B(x[j,k], r_1)} |P_-F_A| \ge z_4 \cdot \xi^*/r^2. \tag{4.32}$$

The number z_4 is chosen so that

$$\sum_k{}' \|P_-F_A\|_{L^2; B(x[j,k], r_1)}^2 \le q/2 \cdot \|P_-F_A\|_{L^2; B(x[j], 3r/4)}^2, \tag{4.33}$$

where $\sum'$ means to sum over those indices k for which (4.32) fails to hold.

Lemma 4.4 asserts that for all r_1 sufficiently small, the fact that (4.32) holds for $[A,h] \in U_1$ and for indices (j,k) implies that for all $x \in B(x[j,k], r_1)$,

$$|P_-F_A|(x) > \tfrac{1}{4} \cdot z_4 \cdot \xi^*/r^2, \tag{4.34}$$

and for any pair $\{x,y\} \subset B(x[j,k], r_1)$,

(4.35)
$$r^2 \cdot |h[x](A,h)^{-1} \cdot P_-F_A(x) \cdot h[x](A,h) - h[y](A,h)^{-1} \cdot P_-F_{A_0}(y) \cdot h[y](A,h)| \leq z_0 \cdot \xi^* \cdot r_1/r.$$

Here, $h(x) \in P|_x$ is obtained from h_0 by parallel transport using the connection A along the path $\varphi[x]$.

Lemma 4.10 and (4.29)–(4.35) imply that for $[A,h] \in U_1$ and those indices (j,k) where (4.34) holds, there exists an index $\beta \in \{1, \cdots, m\}$ which is such that

$$\begin{aligned} \|P_-F_{A'(w)}\|^2_{L^2} \leq \|P_-F_A\|^2_{L^2} &- c_2 \cdot \alpha^2 \cdot \xi^* \cdot r_1^2/(m^2 \cdot r^2) \\ &+ z_0 \cdot (R^4 \cdot \alpha^4 + \alpha^4 + (\alpha^2/R) \cdot \xi^* \cdot r_1^2/(m^2 \cdot r^2)), \end{aligned} \tag{4.36}$$

where $c_2 > 0$ is a constant which is fixed independently of the data at hand. Indices (j,k,β) such that

$$\begin{aligned} \|P_-F_{A'(w)}\|^2_{L^2} \leq \|P_-F_A\|^2_{L^2} &- \tfrac{1}{2} \cdot c_2 \cdot \alpha^2 \cdot \xi^* \cdot r_1^2/(m^2 \cdot r^2) \\ &+ z_0 \cdot (R^4 \cdot \alpha^4 + \alpha^4 + (\alpha^2/R) \cdot \xi^* \cdot r_1^2/(m^2 \cdot r^2)) \end{aligned} \tag{4.37}$$

will be called "good" indices for $[A,h]$. (In (4.37), $\frac{1}{2} \cdot c_2$ replaces c_2 in (4.36).)

Choose

$$\alpha^2 \equiv s[j,k,\beta] \cdot \xi^* \cdot r_1^2/(m^2 \cdot r^2 \cdot R^8). \tag{4.38}$$

Here $s[j,k,\beta]\colon U_1 \to (0,1]$ is a continuous function which will be determined shortly. Independent of the data at hand, one can adjust R and r_1 so that when (j,k,β) is a good index for $[A,h]$, then (4.37) implies

$$\|P_-F_{A'(w)}\|^2_{L^2} \leq \|P_-F_A\|^2_{L^2} - c_4 \cdot s[j,k,\beta] \cdot \xi^{*2} \cdot r_1^4/r^4, \tag{4.39}$$

where $c_4 > 0$ is a constant which is defined independently of all the data at hand; it depends just on the Lie group G and the Riemannian manifold M.

If (j,k,β) is not a good index for $[A,h]$, then one has

$$\|P_-F_{A'(w)}\|^2_{L^2} \leq \|P_-F_A\|^2_{L^2} + c_5 \cdot s[j,k,\beta] \cdot \xi^{*2} \cdot r_1^4/r^4. \tag{4.40}$$

Here, again, the constant c_4 is independent of the data at hand.

It remains to choose the functions $\{s[j,k,\beta]\colon U_1 \to (0,1)\}$. For a given $[A,h] \in U_1$ and index (j,k), choose $s[j,k,\beta](A,h) \equiv 1$ on the compact set in U_1 where (4.36) is satisfied. Choose $s[j,k,\beta](A,h) <<< 1$ on the open set in U_1 where (4.37) is not satisfied, and smoothly interpolate between. One can leave the precise choice of s, where (4.37) is not obeyed, somewhat arbitrary; it will be convenient to exploit this freedom in §7 when the behavior of the self-dual connections under this map under construction is considered.

Now, observe that for any index (j, k, β), and for any $[A, h] \in U_1$.

$$(4.41)\qquad \|P_-F_A\|^2_{L^2;B(x[j,k,\beta],r_1/m)} \le z_0 \cdot \xi^{*2} \cdot r_1^4/r^4.$$

Here z_0 is independent of the data at hand. Together, (4.28), (4.33), and (4.41) imply that for any index j and any $[A, h] \in U_1(\delta/2, j)$

$$(4.42)\qquad \sum_{(k,\beta):\ [4.36] \text{ is obeyed}} (\xi^{*2} \cdot r_1^4/r^4) \ge q_1 \cdot \|P_-F_A\|^2_{L^2;B(x[j],3r/4)},$$

with $q_1 > 0$ being independent of the data at hand. Meanwhile, (4.33) implies that for any index j and any $[A, h] \in U_1$,

$$(4.43)\qquad \sum_{(k,\beta)} (\xi^{*2} \cdot r_1^4/r^4) \le z_0 \cdot \|P_-F_A\|^2_{L^2;B(x[j],3r/4)},$$

with z_0 being independent of the data at hand.

Let J denote the total number of indices $\{(j, k, \beta)\}$. Let (k_0, η) denote the characteristic classes of the bundle P. By gluing, as directed above, at all of the points $\{x[j, k, \beta]\}$ and for each $[A, h] \in U_1$, one obtains a map $\Theta\colon U_1 \to \mathfrak{B}(k_0 + J \cdot c(G), \eta)$ which is continuous due to Lemmas 4.4 and 4.6 and which is homotopic to the J-fold composition of the map θ_+ of (4.2). Furthermore, Lemma 4.9 plus (4.27), (4.28) and (4.39)–(4.43) imply

$$(4.44)\qquad \sup_{[A,h]\in U_1} \|P_-F_{\Theta(A,h)}\|^2_{L^2} < (1 - z) \cdot \delta,$$

where $z > 0$ depends only on the Lie group G and the manifold M. This last equation proves Proposition 4.2.

5. The topology of $\mathfrak{B}'_\epsilon(k_0, \eta)$ for small ϵ

Let (k_0, η) be admissible characteristic classes for a principal G-bundle over M. As in Proposition 4.2, fix $\epsilon > 0$ and let $\mathfrak{F}$ denote a homotopy invariant family of compact subsets of $\mathfrak{B}'(k_0, \eta)$ $(\mathrm{rel}\, \mathfrak{B}'_\epsilon(k_0, \eta))$, and for each integer $j > 0$, let $\mathfrak{F}(j)$ denote the homotopy equivalent, homotopy invariant family of compact subsets of $\mathfrak{B}'(k_0 + j \cdot c(G), \eta)$ $(\mathrm{rel}\, \mathfrak{B}'_{2\epsilon}(k_0 + j \cdot c(G), \eta))$ as described in Proposition 4.2. Proposition 4.2 asserts that by taking the integer j sufficiently large, there will exist $U \in \mathfrak{F}(j)$ for which the supremum over $[A, h] \in U$ of $\|P_-F_A\|_{L^2}$ will be less than $2 \cdot \epsilon$. The object of this section is to study the topology of $\mathfrak{B}'_\epsilon(k, \eta)$ relative to $\mathfrak{M}'(k, \eta)$. In the next two sections, the large k behavior of this relative topology will be studied.

This study of the topology of $\mathfrak{B}'_\epsilon(k,\eta)$ relative to $\mathfrak{M}'(k,\eta)$ uses for the most part modified versions of the techniques which were introduced in [20] and in [21]. The strategy here is to find obstructions to pushing a given compact set $U \subset \mathfrak{B}'_\epsilon(k,\eta)$ onto $\mathfrak{M}'$. The obstruction to pushing a given point onto $\mathfrak{M}'$ can be interpreted as the nonvanishing at the point in question of a canonical section of a particular, local vector bundle. The vector bundle in question is defined over an open set from an open cover of $\mathfrak{B}'_\epsilon(k,\eta)$. Taken together, the vector bundles over the various open sets from the open cover do not make a globally defined vector bundle; rather, they define a virtual bundle in K-theory. This "bundle" will be called the obstruction bundle; it and the canonical section are described in this section (see also §3 of [21]). The zero set of the canonical section is described in §6.

To begin, fix $\epsilon > 0$ and consider a compact set $U \subset \mathfrak{B}'(k,\eta)$ such that the supremum over $[A,h] \in U$ of $\|P_-F_A\|_{L^2}$ is less than ϵ. Preliminary deformations of such a U must be made to obtain $U' \subset \mathfrak{B}'_\epsilon(k,\eta)$ which obeys a priori estimates. The following lemma provides the necessary deformation.

Lemma 5.1. *Let M be a compact, oriented Riemannian 4-manifold, and let G be a compact Lie group. There exists $\epsilon > 0$ and $\varsigma < \infty$ with the following significance: Let $P \to M$ be a principal G-bundle with characteristic classes (k,η). As before, let $\mathfrak{B}'_\epsilon(k,\eta) \equiv \{b \in \mathfrak{B}'(k,\eta) \colon \|P_-F_A\|_{L^2} < \epsilon\}$. Given $\delta > 0$, there exists a smooth, $\mathfrak{a}$-decreasing homotopy $H_\delta \colon [0,1] \times \mathfrak{B}'_\epsilon(k,\eta) \to \mathfrak{B}'_\epsilon(k,\eta)$ which is constant on $\mathfrak{M}'(k,\eta)$, and which has the property that when $[A_1,h_1]$ denotes $H_\delta(1,[A,h])$ for $[A,h] \in \mathfrak{B}'_\epsilon(k,\eta)$, then*

$$\|P_-F_{A_1}\|_{L^4} + \sup_{x\in M} \|\mathrm{dist}(\cdot,x)^{-2} \cdot P_-F_{A_1}\|_{L^1} \le \varsigma \cdot \|P_-F_A\|_{L^2} + \delta.$$

This lemma will be proved shortly.

In the case where $M = S^4$ with its standard metric, the homotopy of Lemma 5.1 is the homotopy in §3 of [20]; it converges as $t \to \infty$ and provides a retraction of $\mathfrak{B}'_\epsilon(k,\eta)$ onto $\mathfrak{M}'(k,\eta)$. This is Proposition 3.1 in [20]. For the general case, such a retraction may not exist; for example, if the intersection form on $H_2(M)$ is indefinite. (K. Uhlenbeck [13] established that for a suitably generic metric on TM, the moduli space $\mathfrak{M}'(k,\eta)$ is a smooth submanifold of $\mathfrak{B}'(k,\eta)$ whenever (k,η) are admissible as characteristic classes for a principal G-bundle over M. When such is the case, there will be a tubular neighborhood of $\mathfrak{M}'(k,\eta)$ in $\mathfrak{B}'(k,\eta)$ which retracts onto $\mathfrak{M}'(k,\eta)$. However, the functional $\mathfrak{a}$ may not be bounded away from zero on the compliment of such a neighborhood; see Lemma 5.4, below.)

In the case where $\mathfrak{B}'_\epsilon(k,\eta)$ does not retract onto $\mathfrak{M}'(k,\eta)$, it is necessary to generalize certain constructions which were introduced in [21]. These generalizations occupy the remainder of this section. The strategy which was employed in [21] will be used here, essentially unchanged. However, the estimates on which the successful application of that strategy were based must be refined for application here. Estimates are required which are independent of the Pontrjagin class of the bundle P. The estimates which are developed in §3 of [21] do not have this property.

To begin, define, for $\epsilon > 0$, the space

$$\mathfrak{b}'_\epsilon(k,\eta) \equiv \left\{[A,h] \in \mathfrak{B}'_\epsilon(k,\eta)\colon \|P_-F_A\|_{L^2} + \sup_{x\in M} \|\mathrm{dist}(\cdot,x)\cdot P_-F_A\|_{L^1} \le \epsilon\right\}.$$

Lemma 5.1 asserts that $\epsilon_0 > 0$ exists which depends only on G and the Riemannian metric and which is such that $\mathfrak{B}'_\epsilon(k,\eta)$ retracts onto $\mathfrak{b}'_{\varsigma\cdot\epsilon}(k,\eta)$ when $\epsilon < \epsilon_0$. For such ϵ, fix $\mu > 0$, and define the subset $\mathfrak{U}(\mu) \subset \mathfrak{b}'_\epsilon(k,\eta)$ to be the set of orbits $[A]$ such that μ is not an eigenvalue of the unbounded operator $P_-d_A(P_-d_A^*)$ on $L^2(P_-\Omega^2(\mathrm{Ad}\,P))$. By standard perturbation arguments [14], there is a smooth map $\Pi(\mu)[\cdot]$ of $\mathfrak{U}(\mu)$ into the space of bounded, linear operators on $L^2(P_-\Omega^2(\mathrm{Ad}\,P))$ that sends $[A,h] \in \mathfrak{U}(\mu)$ to the operator $\Pi(\mu;A)$ on $L^2(P_-\Omega^2(\mathrm{Ad}\,P))$ which gives the L^2-orthogonal projection onto the span of the eigenvectors of $P_-d_A(P_-d_A^*)$ with eigenvalue less than μ. Note that the dimension of the image of $\Pi(\mu;A)$ in $L^2(P_-\Omega^2(\mathrm{Ad}\,P))$ is bounded a priori by Proposition A.1.

Fix $\mu_0 \in (0,1/2]$. Since there may be spectral flow for $P_-d_A(P_-d_A^*)$ as $[A,h]$ varies through $\mathfrak{b}'_\epsilon(k,\eta)$, one cannot, in general, guarantee that $\mathfrak{b}'_\epsilon(k,\eta)$ can be covered by only one set $\mathfrak{U}(\mu)$ for $\mu \in (\mu_0, 2\cdot\mu_0)$. However, due to Proposition A.1, there exists a number $n_0 < \infty$ which depends only on the Riemannian metric on M, and which is such that the number of eigenvalues of $P_-d_A(P_-d_A^*)$ which lie in $(\mu_0, 2\cdot\mu_0)$ is bounded a priori by n_0. This means that $\mathfrak{b}'_\epsilon(k,\eta)$ can be covered by n_0+1 sets $\{\mathfrak{U}(\mu_i)\}$ with $\mu_i \in (\mu_0, 2\cdot\mu_0)$. Let $\{\psi[\mu_0\colon \mu_i]\colon \mathfrak{U}(\mu_i) \to [0,1]\}$ be a smooth partition of unity for the cover of $\mathfrak{b}'_\epsilon(k,\eta)$ by $\{U(\mu_i)\}$.

When $[A,h] \in \mathfrak{b}'_\epsilon(k,\eta)$, try, as in [21], to define a self-dual orbit $[A+a(A),h] \in \mathfrak{M}'(k,\eta)$ where $a(A) \equiv P_-d_A^*v(A)$, and where $v(A) \in L^2_3(P_-\Omega^2(\mathrm{Ad}\,P))$ solves the equation

$$\begin{aligned}(5.1)\qquad & P_-d_A(P_-d_A^*v)\\ &\quad + \sum_i \psi[\mu_0;\mu_i]\cdot(1-\Pi(\mu_i;A))\cdot P_-(P_-d_A^*v\wedge P_-d_A^*v + F_A) = 0.\end{aligned}$$

Lemma 5.2. *Let M be a compact, oriented, Riemannian 4-manifold, and let G be a compact Lie group. There exist constants $\delta_0 > 0$ and $z_0 \in [1,\infty)$ which have the following significance: Let $P \to M$ be a principal G-bundle with characteristic classes (k,η), and let $\pi\colon (\mathfrak{A}(P)\times P|_{x_0}) \to \mathfrak{B}'(k,\eta)$ denote* the canonical projection. For $\epsilon \in (0, \epsilon_0]$, let $(A, h) \in \pi^{-1}\mathfrak{b}'_\epsilon(k, \eta)$. Set $\mu_0 \equiv z_0 \cdot \epsilon^{1/4}$ in (5.1); then this equation has a unique, small solution, $v(\mu_0; A) \in L^2_3(P_-\Omega^2(\mathrm{Ad}\, P))$. Furthermore, the assignment of $v(\mu_0; A)$ to (A, h) defines a smooth, $\mathfrak{G}(P)$-equivariant map from $\pi^{-1}\mathfrak{b}'_\epsilon(k, \eta)$ to $L^2_3(P_-\Omega^2(\mathrm{Ad}\, P))$.

This lemma will also be proved shortly.

Given $\epsilon \in (0, \epsilon_0]$, Lemma 5.2 defines a smooth section over $\mathfrak{b}'_\epsilon(k, \eta)$ of the vector bundle $\mathfrak{V} \equiv \{[A, h, L^2_3(P_-\Omega^2(\mathrm{Ad}\, P))]: [A, h] \in \mathfrak{b}'_\epsilon(k, \eta)\}$. This section, $\mathfrak{s}_{\mu_0}$, sends $[A, h]$ to

(5.2)

$$\mathfrak{s}_{\mu_0}([A, h]) \equiv [A, h, \Pi(2 \cdot \mu_0(\epsilon); A) \cdot P_-(P_- d^*_A v(\mu_0; A) \wedge P_- d^*_A v(\mu_0; A) + F_A)].$$

The zero set of this section defines the moduli space $\mathfrak{M}'(k, \eta)$ of self-dual connections on P. Indeed, if $[A, h] \in \mathfrak{b}'_\epsilon(k, \eta)$, then $[A + P_- d^*_A v(\mu_0; A), h] \in \mathfrak{B}'(k, \eta)$ is self-dual if and only if $\mathfrak{s}_{\mu_0}([A, h]) = 0$. Furthermore, the assignment of $(t, [A, h]) \in [0, 1] \times \mathfrak{b}'_\epsilon(k, \eta) \to [A + t \cdot P_- d^*_A v(\mu_0; A), h] \in \mathfrak{B}'(k, \eta)$ defines a homotopy of $\mathfrak{b}'_{\epsilon/2}(k, \eta)$ in $\mathfrak{B}'(k, \eta)$ which fixes $\mathfrak{M}'(k, \eta)$ and which deforms $\mathfrak{s}_{\mu_0}(\cdot)^{-1}(0)$ onto $\mathfrak{M}'(k, \eta)$. Since $\mathfrak{b}'_{\epsilon/2}(k, \eta)$ is open in $\mathfrak{B}'(k, \eta)$, it is a straightforward matter to extend this homotopy to a homotopy of $\mathfrak{B}'(k, \eta)$. Thus, Lemma 5.2 has the following corollary:

Lemma 5.3. *Let M be a compact, oriented, Riemannian 4-manifold, and let G be a compact Lie group. There exists constants $\epsilon_0 > 0$, $z_0 < \infty$, and these have the following significance: Let $P \to M$ be a principal G-bundle with characteristic classes (k, η). For $\epsilon \in (0, \epsilon_0]$, set $\mu_0 \equiv \mu_0(\epsilon) = z_0 \cdot \epsilon^{1/4}$ in* (5.1) *and* (5.2). *There is a smooth homotopy $\Phi: [0, 1] \times \mathfrak{B}'(k, \eta) \to \mathfrak{B}'(k, \eta)$ which fixes $\mathfrak{M}'(k, \eta)$, which maps $\{1\} \times \mathfrak{b}'_{\epsilon/2}(k, \eta)$ into $\mathfrak{b}'_{z_0 \cdot \epsilon}(k, \eta)$, and which maps $\{1\} \times (\mathfrak{s}_{\mu_0}(\cdot)^{-1}(0) \cap \mathfrak{b}'_{\epsilon/2}(k, \eta))$ onto $\mathfrak{M}'(k, \eta)$.*

This lemma will also be proved shortly.

As previously noted, for a suitably generic metric on TM all of the moduli spaces $\mathfrak{M}'(k, \eta)$ are either empty, or smooth manifolds [13]. For such metrics, one expects that a tubular neighborhood of a nonempty $\mathfrak{M}'(k, \eta)$ will retract onto $\mathfrak{M}'(k, \eta)$. To begin the construction of such a tubular neighborhood, observe that the assignment of $[A, h]$ in $\mathfrak{B}'(k, \eta)$ to the smallest eigenvalue, $E_0[A]$, of the operator $P_- d_A (P_- d_A)^*$ defines a continuous map $E_0[\cdot]: \mathfrak{B}'(k, \eta) \to [0, \infty)$. For the generic metrics in [13], the assignment of $E_0[A]$ to $[A, h] \in \mathfrak{M}'(k, \eta)$ defines a smooth map $E_0[\cdot]: \mathfrak{M}'(k, \eta) \to (0, \infty)$. This fact and the following lemma complete the tubular neighborhood construction.

Lemma 5.4. *Let M be a compact, oriented, Riemannian 4-manifold, and let G be a compact Lie group. There exists a continuous function $z(\cdot): [0, \infty) \to [0, \infty)$ which maps 0 to 0 and which has the following additional property:*

Let $P \to M$ be a principal G-bundle with characteristic classes (k, η). Let $\mathfrak{N}'(k,\eta) \equiv \{[A,h] \in \mathfrak{B}'(k,\eta)\colon \|P_-F_A\|_{L^2} \le z(E_0[A])\}$. There is a continuous retraction of $\mathfrak{N}'(k,\eta)$ onto $\mathfrak{M}'(k,\eta)$ which fixes $\mathfrak{M}'(k,\eta)$.

The remainder of this section contains the proofs of Lemmas 5.1–5.4.

Proof of Lemma 5.1. The proof of this lemma mimics the proof of Proposition 3.1 of [20]. To begin, let $P \to M$ be a principal G-bundle with characteristic classes (k,η). Define $\mathfrak{A}_\epsilon \equiv \{A \in \mathfrak{A}(P)\colon \|P_-F_A\|_{L^2} < \epsilon\}$. For any $m > 0$ and $A \subseteq \mathfrak{A}_\epsilon$, there exists a unique $u(A) \subseteq L^3_2(P_-\Omega^z(\operatorname{Ad} P))$ which solves the differential equation

$$P_-d_A(P_-d_A^*u) + m \cdot u = -P_-F_A. \tag{5.3}$$

Furthermore, $u(A)$ varies smoothly with A and defines a $\mathfrak{G}(P)$-equivariant map from $\mathfrak{A}_\epsilon$ into $L^3_2(P_-\Omega^2(\operatorname{Ad} P))$.

Let $a(A) \equiv P_-d_A^*u(A) \in L^2_2(\Omega^1(\operatorname{Ad} P))$. The assignment of $A \in \mathfrak{A}_\epsilon$ to $a(A)$ defines a smooth, $\mathfrak{G}(P)$-equivariant map. Thus, by push-forward, the assignment of $[A,h] \in \mathfrak{B}'(k,\eta)_\epsilon$ to $[A,h,a(A)] \in (\mathfrak{A}_\epsilon \times_{G(P)} L^2_2(\Omega^1(\operatorname{Ad} P)))$ defines a smooth vector field on $\mathfrak{B}'(k,\eta)_\epsilon$. The homotopy in question will be obtained by integrating this vector field.

To study the integral curves of the vector field, above, fix $A \in \mathfrak{A}_\epsilon$. Then standard short-time existence theory provides $T(A) > 0$, and a unique, smooth map $\Phi(\cdot, A)\colon [0,T) \to A_\epsilon$ which satisfies $\Phi(0,A) = A$, and

$$\frac{\partial}{\partial t}\Phi(\cdot, A) \equiv a(\Phi(\cdot, A)) \tag{5.4}$$

(see Theorem 4.1.13 in [1]). The fact that Φ is unique implies that $\Phi(\cdot, g\cdot A) \equiv g \cdot \Phi(\cdot, A)$ for all $g \in \mathfrak{G}(P)$. This short time existence theory provides an open neighborhood of A, $\mathfrak{U} \subset A_\delta$ which is such that $\Phi(\cdot, A')$ is defined on $(0, T(A)]$ for all $A' \in \mathfrak{U}$; and $\mathfrak{U}$ is such that the assignment of $(t, A') \in [0,T) \times \mathfrak{U}$ to $\Phi(t, A') \in A_\delta$ defines a smooth map.

Lemma 5.5. *Let M be a compact, oriented Riemannian 4-manifold, and let G be a compact Lie group. There exists $\epsilon > 0$ and $\varsigma < \infty$ with the following significance: Let $P \to M$ be a principal G-bundle, and let $A \in \mathfrak{A}_\epsilon(P)$. Then, the curve $\Phi(\cdot, A)$ of* (5.4) *has a unique extension to a map from $[0,\infty)$ into $\mathfrak{A}_\epsilon$ which obeys* (5.4) *for all $t \in [0,\infty)$. Furthermore, $\Phi(\cdot,\cdot)$ defines a smooth, $\mathfrak{G}(P)$-equivariant map from $[0,\infty) \times \mathfrak{A}_\epsilon \to \mathfrak{A}_\epsilon$.*

This lemma will be proved shortly.

Given Lemma 5.5, the proof of Lemma 5.1 is completed by observing that on an interval in $[0,\infty)$ where $\Phi(\cdot, A)$ is defined, the curvature of $\Phi(\cdot, A)$

evolves according to

$$\frac{\partial}{\partial t} P_- F_{\Phi(\cdot,A)} = -P_- F_{\Phi(\cdot,A)} - u(A). \tag{5.5}$$

This last equation implies that the L^2-norm of $P_- F_{\Phi(\cdot,A)}$ is a nonincreasing function of $t \in [0,T)$ which is strictly decreasing unless A is a critical point of $\mathfrak{a}$; indeed,

$$\begin{aligned} &\frac{\partial}{\partial t} \|P_- F_{\Phi(\cdot,A)}\|_{L^2}^2 \\ &\qquad \equiv -2 \cdot \langle P_- F_{\Phi(\cdot,A)}, (P_- d_A (P_- d_A)^* + m)^{-1} \cdot P_- F_{\Phi(\cdot,A)} \rangle_{L^2}. \end{aligned} \tag{5.6}$$

Note also that if $x \in M$ is fixed, then

$$\begin{aligned} &\frac{\partial}{\partial t} \|\mathrm{dist}(\cdot,x)^{-2} \cdot P_- F_{\Phi(\cdot,A)}\|_{L^1} \\ &\qquad \leq -\|\mathrm{dist}(\cdot,x)^{-2} \cdot P_- F_{\Phi(\cdot,A)}\|_{L^1} - m \cdot \|\mathrm{dist}(\cdot,x)^{-2} \cdot u(\Phi(\cdot,A))\|_{L^1} \\ &\qquad \leq -\|\mathrm{dist}(\cdot,x)^{-2} \cdot P_- F_{\Phi(\cdot,A)}\|_{L^1} + z \cdot m \cdot \|u(\Phi(\cdot,A))\|_{L^4}, \end{aligned} \tag{5.7}$$

where z is a constant which only depends on the Riemannian metric.

Equation (5.3) can be used to estimate the L^4 norm of $u(\Phi(\cdot,A))$. Let $A' \in \mathfrak{A}_\epsilon$ be given. Via the Weitzenboch formula (cf. Appendix C in [13]) and an integration by parts,

$$\|\nabla_{A'} u\|_{L^2}^2 + m \cdot \|u\|_{L^2}^2 + \langle u, \mathfrak{W} \cdot u \rangle_{L^2} + \langle u, \{P_- F_{A'}, u\} \rangle_{L^2} = \langle u, P_- F_{A'} \rangle_{L^2}, \tag{5.8}$$

where $\mathfrak{W} \cdot \in C^\infty(\mathrm{End}(P_- \bigwedge^2 T^*))$ is a component of the Riemannian curvature tensor, while, with respect to a local basis for $P_- \bigwedge^2 T^*$, $\{P_- F_{A'}, u\}$ is a specific linear combination of the commutator of the components of $P_- F_{A'}$ and of u (see Appendix C in [13]).

Take $m \equiv 4 \cdot \sup_M |\mathfrak{W} \cdot|$. Then (5.8) provides ϵ_0 which depends only on the Riemannian metric on TM and which is such that if $\epsilon < \epsilon_0$, then

$$\|\nabla_{A'} u\|_{L^2}^2 + m \cdot \|u\|_{L^2}^2 \leq 2 \cdot \langle P_- F_{A'}, (P_- d_{A'} (P_- d_{A'})^* + m)^{-1} \cdot P_- F_{A'} \rangle_{L^2}. \tag{5.9}$$

With Kato's inequality and the $L_1^2 \to L^4$ Sobolev space inclusion ((3.4) of [22]), (5.9) yields the bound

$$\|u(A')\|_{L^4}^2 \leq z \cdot \langle P_- F_{A'}, (P_- d_{A'} (P_- d_{A'})^* + m)^{-1} \cdot P_- F_{A'} \rangle_{L^2}, \tag{5.10}$$

with z depending only on the Riemannian metric on TM.

Using (5.6) and (5.10), one can integrate (5.7) to conclude that

$$\begin{aligned} \|\mathrm{dist}(\cdot,x)^{-2} \cdot P_- F_{\Phi(t,A)}\|_{L^1} &\leq e^{-t} \cdot \|\mathrm{dist}(\cdot,x)^{-2} \cdot P_- F_A\|_{L^1} \\ &\qquad + z \cdot \|P_- F_{\Phi(t,A)}\|_{L^2}. \end{aligned} \tag{5.11}$$

A similar argument shows that

$$\|P_- F_{\Phi(t,A)}\|_{L^4} \leq e^{-t} \cdot \|P_- F_A\|_{L^4} + z \cdot \|P_- F_{\Phi(t,A)}\|_{L^2}. \tag{5.12}$$

Make the homotopy $H_\epsilon(\cdot,\cdot)$ of Lemma 5.1 send $(t,[A,h])$ to

$$[h, \Phi(\ln(4\cdot(\|\mathrm{dist}(\cdot,x)^{-2}\cdot P_-F_A\|_{L^1} + \|P_-F_A\|_{L^4}+1)/\epsilon), A)].$$

Proof of Lemma 5.5. This mimics the argument which proves Proposition 3.1 of [20]. First of all, the reader can readily check that there exists $\epsilon_0 > 0$ which depends only on the Riemannian metric, and which is such that when $\epsilon < \epsilon_0$ and when $A \in \mathfrak{A}_\epsilon$, then $u(A)$ obeys the conclusions of Lemmas 3.5–3.7 of [20] with constants which depend only on the Riemannian metric.

Now, let $\chi(\cdot)\colon [0,\infty) \to [0,1]$ be a smooth function which vanishes on $[3/2,\infty)$, and which is identically 1 on $[0,1]$. For $x \in M$ and $r > 0$, let $\chi[x,r](\cdot) \equiv \chi(\mathrm{dist}(\cdot,x)/r)$. For $A \in \mathfrak{A}_\epsilon$, suppose that $A(\cdot) \equiv \Phi(\cdot, A)$ is defined on $[0,T)$. Analogous to (3.16) in [20], one has (using (5.12)) the bound

$$\left|\frac{\partial}{\partial t}\|\chi[x,r]\cdot F_{A(t)}\|_{L^2}\right| \le z([A]), \tag{5.13}$$

where $z([A])$ depends only on the starting orbit $[A] \equiv [A(0)]$.

When $T < \infty$, this last inequality implies, using a variation of the proof of Lemma 3.10 in [20], that the sequence of connections $\{A(t)\}_{t\to T}$ is a Cauchy sequence with respect to the L^2_1-topology on $\mathfrak{A}_\epsilon$. The convergence of $\{A(t)\}_{t\to T}$ in stronger norms is obtained in a straightforward way by taking covariant derivatives of (5.5); the details are omitted (see Lemma 3.11 in [20] for an example).

Thus, when $T < \infty$, the map which sends $t \in [0,T)$ to $\Phi(t,A)$ has a unique extension to a continuous map from $[0,T]$ into $\mathfrak{A}_\epsilon$. By the local existence result, the map has a unique extension to $[0,T+\sigma)$ for some $\sigma > 0$. Standard arguments yield the remaining assertions of the lemma.

To prove Lemma 5.2, one must exploit the fact that for $\mu > 0$ and for any connection A, the operator $H_A \equiv P_-d_A(P_-d_A^*)$ has bounded inverse on $(1-\Pi(\mu;A))\cdot L^2$. Thus, if v solves (5.1), then v solves the fixed point equation

$$\begin{aligned} v = -H_A^{-1}\cdot(1-\Pi(\mu_0;A))\cdot\sum_i \psi[\mu_0;\mu_i]\cdot(1-\Pi(\mu_i;A))& \\ \cdot P_-(P_-d_A^*v\wedge P_-d_A^*v + F_A).& \end{aligned} \tag{5.14}$$

Equation (5.14) will be analyzed as a fixed point equation on the Banach space

$$\begin{aligned} U(\mu;A) \equiv \Big\{ u \in (1-\Pi(\mu_0;A))\cdot L^2(P_-\Omega^2(\mathrm{Ad}\,P))\colon& \\ |u|_U^2 \equiv \|\nabla_A u\|_{L^2}^2 + \|u\|_\infty^2 + \sup_{x\in M}\|\mathrm{dist}(\cdot,x)^{-2}\cdot|\nabla_A u|^2\|_{L^1} < \infty \Big\}.& \end{aligned}$$

It is useful to define a second Banach space W to be the completion of $L^2(P_-\Omega^2(\operatorname{Ad} P))$ with the norm $|w|_W \equiv \|w\|_{L^1} + \sup_{x\in M} \|\operatorname{dist}(\cdot,x)^{-2}\cdot w\|_{L^1}$. The utility of these spaces is due to

Lemma 5.6. *Let M be a compact, oriented, Riemannian 4-manifold, and let G be a compact Lie group. There exists constants $\epsilon_1 > 0$ and $z_0 < \infty$ which have the following significance: Let $P \to M$ be a principal G-bundle with characteristic classes (k,η) and let $[A,h] \in \mathfrak{b}_{\epsilon_1}(k,\eta)$. For $\mu > 0$, H_A^{-1} defines a bounded map from $(1-\Pi(\mu;A))\cdot L^2(P_-\Omega^2(\operatorname{Ad} P)) \cap W$ to $U(\mu;A)$ and $|H_A^{-1}w|_U \leq z_0\cdot\mu^{-1}\cdot|w|_W$.*

Proof of Lemma 5.6. First, suppose that w is smooth, and in $(1-\Pi(\mu;A))\cdot L^2(P_-\Omega^2(\operatorname{Ad} P))$, then standard arguments provide a unique $u \in L^2_3(P_-\Omega^2(\operatorname{Ad} P))$ which satisfies

$$P_-d_A(P_-d_A^* u) = w. \tag{5.15}$$

Contract both sides of this equation with u and integrate over M to derive the bound

$$\|u\|_{L^2}^2 \leq \mu^{-1}\cdot\|u\|_\infty\cdot\|w\|_{L^1}. \tag{5.16}$$

The Weitzenboch formula (see Appendix C of [13]) implies that

$$\tfrac{1}{2}\cdot d^*d|u|^2 + |\nabla_A u|^2 + (u,\mathfrak{W}\cdot u) + (u,\{P_-F_{A'},u\}) = (u,w). \tag{5.17}$$

Integrate both sides of the preceding equation over M. Using (3.4), one finds $\epsilon_0 > 0$ and $z_0 < \infty$ which depend only on the Riemannian metric of M and which are such that when $\epsilon < \epsilon_0$, then

$$\|\nabla_A u\|_{L^2}^2 \leq z_0\cdot\mu^{-1}\cdot\|u\|_\infty\cdot\|w\|_{L^1}. \tag{5.18}$$

Next, fix $x \in M$ and let $G(\cdot,x) \in C^\infty(M\backslash\{x\})$ denote the Green's function for d^*d with pole at x. There exists $r_0 > 0$ which depends only on the metric on M and is such that $\frac{1}{2}\cdot\operatorname{dist}(y,x)^{-2} < G(y,x) < 2\cdot\operatorname{dist}(y,x)^{-2}$ whenever $\operatorname{dist}(y,x) < r_0$.

Multiplying both sides of (5.17) by $G(\cdot,x)$, and then integrate the result over M. The following bound is obtained:

$$\begin{aligned} &|u|^2(x) + \|\operatorname{dist}(y,x)^{-2}\cdot|\nabla_A u|^2\|_{L^1} \\ &\qquad \leq z_0\cdot(\mu^{-1}\cdot\|u\|_\infty\cdot\|w\|_{L^1} + \|u\|_\infty^2\cdot|P_-F_A|_W + \|u\|_\infty\cdot|w|_W). \end{aligned} \tag{5.19}$$

Here $z_0 < \infty$ depends only on the Riemannian metric. Take the supremum over x In M of the number on the left-hand side of (5.19). One obtains constants $\epsilon_0 > 0$ and $z_0 < \infty$ which only depend on the Riemannian metric and which are such that when $\epsilon < \epsilon_0$,

$$\|u\|_\infty^2 + \sup_{x\in M}\|\operatorname{dist}(\cdot,x)^{-2}\cdot|\nabla_A u|^2\|_{L^1} < z_0\cdot\mu^{-1}\cdot|w|_W^2.$$

With (5.18), this last equation yields the lemma for smooth w. The general case is established by taking limits.

The application of Lemma 5.6 to the fixed point equation (5.14), requires the following additional result:

Lemma 5.7. *Let M be a compact, oriented, Riemannian 4-manifold, and let G be a compact Lie group. There exist constants $\epsilon_1 > 0$ and $z_0 < \infty$ which have the following significance: Let $P \to M$ be a principal G-bundle with characteristic class (k, η) and let $[A, h] \in \mathfrak{b}_{\epsilon_1}(k, \eta)$. For $\mu_0 > 0$, and $u \in C^\infty(P_-\Omega^2(\operatorname{Ad} P))$, let*

$$T[\mu_0; A](u) \equiv H_A^{-1} \cdot (1 - \Pi(\mu_0; A)) \cdot \sum_i \psi[\mu_0; \mu_i] \cdot (1 - \Pi(\mu_i; A)) \cdot P_-(P_- d_A^* u \wedge P_- d_A^* u).$$

Then the assignment of u to $T[\mu_0; A](u)$ extends to define a smooth map $T[\mu_0; A](\cdot)\colon U(\mu_0; A) \to U(\mu_0; A)$ which obeys $|T[\mu_0; A](u)|_U \le z_0 \cdot \mu_0^{-1} \cdot |u|_U^2$.

Proof of Lemma 5.7. Let $\omega \in L_3^2(P_-\Omega^2(\operatorname{Ad} P))$ be an eigenvector of H_A with eigenvalue $\mu(\omega) < 2 \cdot \mu_0$ and with unit L^2-norm. Then it follows from Lemma 5.6 and (3.4) in [20] that

$$|\omega|_U \le z_0 \cdot (1 + \mu_0)^2. \tag{5.20}$$

Now, for smooth u and u', write

$$\begin{aligned}(1 - \Pi(\mu_0; A)) \cdot \sum_i \psi[\mu_0; \mu_i] \cdot (1 - \Pi(\mu_i; A)) \cdot P_-(P_- d_A^* u \wedge P_- d_A^* u') \\ \equiv P_-(P_- d_A^* u \wedge P_- d_A^* u') - \sum \omega_j \\ \cdot \langle \omega_j, \sum_i \psi[\mu_0; \mu_i] \cdot (1 - \Pi(\mu_i; A)) \cdot P_-(P_- d_A^* u \wedge P_- d_A^* u' \rangle_{L^2},\end{aligned} \tag{5.21}$$

where the sum is over an orthonormal basis $\{\omega_j\}$ for the span of the image of $\Pi(2 \cdot \mu_0; A)$. Then, (5.20) and (5.21) imply that

$$\begin{aligned}|(1 - \Pi(\mu)[A]) \cdot P_-(P_- d_A^* u \wedge P_- d_A^* u')|_W \\ \le z_0 \cdot N(\mu; A) \cdot (1 + \mu)^2 \cdot |u|_U \cdot |u'|_U.\end{aligned} \tag{5.22}$$

In this last equation, $N(\mu; A)$ is the dimension of the span of the image of $\Pi(\mu; A)$; this number is bounded a priori in Proposition A.1 by a constant which depends only on μ and the Riemannian metric.

Lemma 5.7 follows immediately from (5.22) and Lemma 5.6.

To analyze (5.14) as a fixed point equation, it is necessary to obtain a bound on the $|\cdot|_U$-norm of the last term on the right-hand side of said equation.

Lemma 5.8. *Let M be a compact, oriented, Riemannian 4-manifold, and let G be a compact Lie group. There exist constants $\epsilon_1 > 0$ and $z_0 < \infty$*

which have the following significance: Let $P \to M$ be a principal G-bundle with characteristic classes (k,η) and let $[A,h] \in b_{\epsilon_1}(k,\eta)$. For $\mu_0 > 0$, and $w \in C^\infty(P_-\Omega^2(\operatorname{Ad} P))$, let

$$\omega[\mu_0; A](w) \equiv H_A^{-1} \cdot (1 - \Pi(\mu_0; A)) \cdot \sum_i \psi[\mu_0; \mu_i] \cdot (1 - \Pi(\mu_i; A)) \cdot w.$$

Then the assignment of w to $\omega[\mu_0; A](w)$ extends to define a smooth map $\omega[\mu_0; A](\cdot)\colon W \to U(\mu_0; A)$ which obeys $|\omega[\mu_0; A](w)|_U \le z_0 \cdot \mu_0^{-1} \cdot |w|_W$.

Proof of Lemma 5.8. This follows from Lemma 5.6; the details are left to the reader.

Proof of Lemma 5.2. Lemmas 5.6–5.8 have established the basic results which allow the use of standard fixed point arguments to prove the existence of a solution to (5.14). In fact, these lemmas provide $\epsilon_0 > 0$, $z_0 < \infty$, and a continuous, decreasing function, $\mu_0(\cdot)\colon [0,\epsilon_0] \to [0,1]$ which maps 0 to 0; these are such that the following is true: For $\epsilon < \epsilon_0$ and for $[A,h] \in \mathfrak{b}'_\epsilon(P)$, there exists a unique $v(\mu_0; A) \in U(\mu_0; A)$ which obeys (5.14) and also satisfies

$$|v(\mu_0, A)|_U \le z_0 \cdot \mu_0^{-1} \cdot |P_- F_A|_W \le \epsilon^{3/4}. \tag{5.23}$$

Lemma 5.9. *There exists a constant $\kappa_0 > 0$ which depends only on the Riemannian metric, and which has the following significance: Let $[A,h]$ and $v(\mu_0; A)$ be as described in the preceding paragraph. Then $v(\mu_0, A) \in L^2_3(P_-\Omega^2(\operatorname{Ad} P))$. Furthermore, if $\int_{B(x,r)} |F_A|^2 < \kappa_0$, for $x \in M$ and $r > 0$, then*

$$\int_{B(x,r/2)} |\nabla_A(P_- d_A^* v)|^2 < z_0 \cdot \mu_0^{-1} \cdot |P_- F_A|_W.$$

Proof of Lemma 5.9. For the a priori estimate, mimic the proof of Lemma 3.7 in [20]. The proof that v is in L^2_3 is a standard bootstrapping argument (see for example [17, Chapter 6]).

The proof of Lemma 5.3 is completed by showing that the assignment of $(A,h) \in \pi^{-1}\mathfrak{b}'_\epsilon(P)$ to $v(\mu_0; A)$ defines a smooth, $\mathfrak{G}(P)$-equivariant map into $L^3_2(P_-\Omega^2(\operatorname{Ad} P))$. This argument is a straightforward application of the contraction mapping principle using the fact that the projectors $\Pi(\mu; A)$ are smoothly varying on the open sets $\mathfrak{U}(\mu)$ (see §8 in [22] for an analogous argument).

Proof of Lemma 5.3. The homotopy in question sends $(t, [A,h]) \in [0,1] \times \mathfrak{b}'_\epsilon(k,\eta)$ to $[A + t\cdot\alpha[A]\cdot P_- d_A^* v(\mu_0; A), h] \in \mathfrak{B}'(k,\eta)$; where $\alpha[\cdot]\colon \mathfrak{b}'_\epsilon(k,\eta) \to [0,1]$ is a smooth, G-equivariant function which is identically one on $\mathfrak{b}'_{\epsilon/2}(k,\eta)$, and which vanishes identically on $\mathfrak{b}'_{3\epsilon/4}(k,\eta)$. For $[A,h] \notin \mathfrak{b}'_\epsilon(k,\eta)$, the homotopy sends $(t,[A,h])$ to $[A,h]$. The only assertion of the lemma which has not yet been verified is the assertion that $z_0 < \infty$ exists which depends only on M and the group G and which is such that for all $[A, h] \in b'_{\epsilon/2}(k, \eta)$, one has $[A + P_-d_A^* v(\mu_0; A), h] \in \mathfrak{b}'_{z_0 \cdot \epsilon}(k, \eta)$. To establish this fact, let $A' \equiv A + P_-d_A^* v(\mu_0; A)$. Then

$$P_-F_{A'} \equiv \sum_i \psi[\mu_0; \mu_i] \cdot \Pi(\mu_i; A) \cdot P_-(P_-d_A^* v \wedge P_-d_A^* v + F_A).$$

The lemma follows from this last equation with (5.20) and (5.22).

Proof of Lemma 5.4. Using Lemmas 5.6–5.9, mimic the proof of Lemma 5.2.

6. The obstruction bundle

This section studies the change with respect to change in Pontrjagin class of the obstructions to self-duality which were defined in the preceding section. Consider a principal G-bundle $P \to M$ with characteristic classes (k, η). Fix $\epsilon > 0$, and focus attention on a compact set $U \subset \mathfrak{b}'_\epsilon(k, \eta)$. Equation (5.2) defines a vector bundle $\mathfrak{V} \to U$, the obstruction bundle, and a canonical section, $\mathfrak{s}_{\mu_0}$, of $\mathfrak{V}$. If $\mathfrak{s}_{\mu_0} \equiv 0$ on U, then U can be deformed continuously into $\mathfrak{M}'(P)$.

Suppose that $\mathfrak{s}_{\mu_0}$ is nonzero on U. Then the following proposition will be invoked.

Proposition 6.1. *Let M be a compact, oriented, Riemannian 4-manifold, and let G be a compact, simple Lie group. There exists $\epsilon_0 > 0$ and $z_0 < \infty$ with the following significance: Let (k, η) be admissible as characteristic classes for a principal G bundle $P \to M$, and suppose that $\mathfrak{M}'(k, \eta)$ is not empty. Let $W \subset \mathfrak{b}'_{\epsilon_0}(k, \eta)$ be a given compact set, and let $U \equiv \mathfrak{M}'(k, \eta) \cup W$. Then there exists an integer $J < \infty$ and for all $J' > J$, a homotopy equivalence $T \equiv T_{J'} \colon \mathfrak{B}'(k, \eta) \to \mathfrak{B}'(k + c(G) \cdot J', \eta)$ which has the following properties: First, T maps $\mathfrak{b}'_{\epsilon_0}(k, \eta) \to \mathfrak{b}'_{z_0 \cdot \epsilon_0}(k + c(G) \cdot J', \eta)$ and U into $\mathfrak{b}'_{z_0 \cdot \epsilon_0/2}(k + c(G) \cdot J', \eta)$, and second, $\mathfrak{s}_{\mu_0(z_0 \cdot \epsilon_0)}|_{T(U)} \equiv 0$, with $\mathfrak{s}_{\mu_0(z_0 \cdot \epsilon_0)}$ as defined in Lemma* 5.3.

The homotopy equivalence, T, is constructed by multiple gluings of the standard self-dual orbit over S^4; $[A_1, h_1] \equiv [A_+, h_1] \in \mathfrak{B}(P_{G_\pm})$, where $[A_+, h]$ and $\mathfrak{B}(P_{G_+})$ are as described in the proof of Proposition 4.1. By a suitable number of gluings, and by an appropriate choice of the gluing parameters, one can construct T with the required properties. Of course, this is the strategy which was developed in [21].

Before beginning the proof proper, it is necessary to investigate how $\mathfrak{s}_{\mu_0(\cdot)}(\cdot)$ is changed by the gluing operation. This investigation occupies the first part of this section, and the results are summarized in Lemma 6.4. The last five parts of this section contains the proof proper of Proposition 6.1.

Part 1: The change in $\mathfrak{s}_{\mu_0(\cdot)}(\cdot)$. To consider the change in $\mathfrak{s}_{\mu_0(\cdot)}(\cdot)$ after gluing, it is necessary to begin by slightly modifying the homotopy of Lemma 4.4 to obtain a priori estimates on orbits in $\mathfrak{b}'_\epsilon(k,\eta)$. The lemma below proves sufficient.

Lemma 6.2. *Let M be a compact, oriented, Riemannian 4-manifold. There exist $r_0 > 0$, $\epsilon_0 > 0$ and $z_0 \in (1,\infty)$, and for each integer $m \geq 0$, there exists $\xi_m < \infty$, and these have the following significance: Let $r \in (0, r_0)$ and $\epsilon \in (0,\epsilon_0)$ and suppose that $J < \infty$ points $\{x(v)\} \subset M$ are fixed such that the set of balls $\{B(x(v),r)\}$ are disjoint. Let $P \to M$ be a principal G-bundle with characteristic classes (k,η), and let $\mathfrak{B}_\epsilon(P;\{x(v),r\}) \equiv \{[A] \in \mathfrak{B}(P)\colon$ (4.7) holds for each ball $B(x(v),r)\}$. There exists a smooth homotopy $\Phi(\{x(v),r\}) \equiv \Phi\colon [0,1] \times \mathfrak{B}'(k,\eta) \to \mathfrak{B}'(k,\eta)$ which maps $[0,1] \times \mathfrak{B}_\epsilon(P;\{x(v),r\}) \to \mathfrak{B}_{z_0\cdot\epsilon}(P;\{x(v),r\})$ and which has the following properties: First, $\Phi(t,[A,h]) = [A,h]$ for all t if $[A,h] \notin \mathfrak{B}_\epsilon(P\colon \{x(v),r\})$. Second, for each $(t,[A]) \in [0,1] \times \mathfrak{B}_\epsilon(P\colon \{x(v),r\})$,*

$$\Phi(t,A) \equiv \left[A + t\cdot\sum_v \beta(\mathfrak{a}(A;x(v),r)/\epsilon)\cdot v'_v(A)\right],$$

where $\mathfrak{a}(A;x,r)$ is defined in (4.7) and $\beta(\cdot)$ is described in Lemma 4.4. Here $[A, v'_v(A)]$ defines a smooth section over $\mathfrak{B}_\epsilon(P;x(v),r)$ of the vector bundle $\mathfrak{A}(P) \times_{\mathfrak{G}(P)} L^2_3(\Omega^1(\operatorname{Ad}P))$ which obeys

(1) *The support of $v'_v(A)$ is in $B(x(v),r)$, and*

$$\|v'_v(A)\|^2_A \leq z_0 \cdot \|\nabla\mathfrak{a}_A\|^2_{A;B(x(v),r)},$$

where $\|\nabla\mathfrak{a}_A\|_{A;B(x(v),r)} \leq z_0\cdot\|P_-F_A\|_{L^2;B(x(v),r)}$ is the norm of the restriction of $\nabla\mathfrak{a}_A$ to the subspace of $L^2_1(\Omega^1(\operatorname{Ad}P))$ with compact support in $B(x(v),r)$.

(2) *For $t \in [0,1]$,*

$$\begin{aligned}\|P_-F_{\Phi(t,A)}\|^2_{L^2} &+ \sup_{x\in M}\|\operatorname{dist}(\cdot,x)^{-2}P_-F_{\Phi(t,A)}\|^2_{L^1}\\ &\leq \|P_-F_A\|^2_{L^2} + \sup_{x\in M}\|\operatorname{dist}(\cdot,x)^{-2}P_-F_A\|^2_{L^1}\\ &\quad + t\cdot z_0\cdot\sum_v\|P_-F_A\|^2_{L^2;B(x(v),r)}.\end{aligned}$$

(3) *For each $[A] \in \mathfrak{B}_{\epsilon/2}(P;\{x(v),r\})$ and for each v and $y \in B(x(v),r/2)$,*

$$|\nabla^{(m)}_{\Phi(1,A)}F_{\Phi(1,A)}|(y) \leq \xi_m\cdot r^{-m-2}\cdot\|F_A\|_{L^2;B(x(v),r)}.$$

(4) *Also, for such $[A]$, and v and y,*

$$|P_-F_{\Phi(1,A)}|(y) + r\cdot|\nabla_{\Phi(1,A)}(P_-F_{\Phi(1,A)})|(y) \leq \xi_0\cdot r^{-2}\cdot\|P_-F_{\Phi(1,A)}\|_{L^2;B(x,r)}.$$

Proof of Lemma 6.2. For each v, construct $v_v(A) \equiv v(A)$ of Lemma 4.4, and then set $v'_v(A) \equiv \beta' \cdot v_v(A)$ where $\beta' \in C^\infty(B(x(v), r); [0, 1])$ is identically 1 on $B(x(v), r/2)$, vanishes identically on $B(x(v), r)\backslash B(x(v), 3r/4)$ and obeys $|d\beta'| \leq 25/r$. The estimates follow from Lemma 4.4. (The estimate on $\|\nabla \mathfrak{a}_A\|_{A;B(x(v),r)}$ in assertion (1) follows from the Bianchi identity after integrating by parts.)

Fix $\epsilon \in (0, \epsilon_0)$, and fix $J < \infty$ points $\{x(v)\} \subset M$ and $r > 0$ such that the set of balls $\{B(x(v), r)\}$ are disjoint. For $[A'_0, h] \in \mathfrak{b}'_\epsilon(k, \eta)$, with $[A_0] \in \mathfrak{B}_{\epsilon/2}(P; \{x(v), r\})$, let $[A_0, h] \equiv \Phi(1, [A'_0, h])$, with Φ as in the preceding lemma. Lemma 6.2 insures that $[A_0, h_0] \in \mathfrak{b}'_{z_0 \cdot \epsilon}(k, \eta)$ with z_0 being independent of b, ϵ and P; it depends only on G and the Riemannian metric.

Let $x_0 \in M$ denote the base point, and let $f_0 \in \mathrm{Fr}\, M|_{x_0}$ be the fixed frame. For each point $x(v)$, choose a smooth path $\varphi(v) \in \mathfrak{P}(x_0)$ from x_0 to $x(v)$. Choose the path to be disjoint from $B(x(v'), r)$ and $\varphi(v')$ for $v' \neq v$. For each v, choose smooth functions $\lambda(v)\colon \mathfrak{B}'(k, \eta) \to (0, 1/8)$ and $t(v)\colon \mathfrak{B}'(k, \eta) \to [8 \cdot \lambda(v)/r, 1)$. Suppose also that for each v, a centered point $[A_1, h_1]_v \in \mathfrak{B}(P_{G_+})$ has been specified. Here, $P_{G_+} \to S^4$ is defined in the proof of Proposition 4.1; as is the unique, centered orbit $[A_1] \equiv [A_+]$.

For fixed v, this data defines data

$$w(v) \equiv [[A_0, h_0], f_0, \varphi(v), s(v) \equiv \lambda(v)/t(v), t(v), [A_1, h_1]_v]$$

which is required for the definition of the connection $A'(w(v))$ of (4.10). This connection is defined on a principal G-bundle, $P' \to M$, with characteristic classes $(k + c(G), \eta)$. The principal bundles P' and P are naturally isomorphic on $M\backslash x(v)$ by an isomorphism which identifies A_0 and $A'(w(v))$ on the compliment of the ball $B(x(v), 4 \cdot \lambda(v)/t(v)) \subset M$.

Since the set of balls $\{B[v] \equiv B(x(v), 4 \cdot \lambda(v)/t(v))\}$ is a set of disjoint balls, the construction of $A'(w(v))$ can be done simultaneously in each ball $B[v]$ to produce a connection $A' \equiv A'(\{w(v)\})$ which is a connection on a principal G-bundle, $P[w] \to M$, with characteristic classes $(k + c(G) \cdot J, \eta)$. It follows from Lemmas 4.4 and 4.7 and (4.16) that J, r, and the parameters $\{\lambda(v), t(v)\}$ can be chosen to insure that $[A', h] \in \mathfrak{b}'_{2z_0 \cdot \epsilon}(k + c(G) \cdot J, \eta)$. Indeed, fix $R \in [1, \epsilon^{-1/4})$ and require that

$$\text{(6.1a)} \qquad t(v) \equiv 16 \cdot \lambda(v)/r, \quad \lambda(v)^4/r^4 < R^4 \cdot \epsilon^2/J, \quad \lambda(v)^2 \leq R \cdot \epsilon/J.$$

Let

$$\text{(6.1b)} \qquad \mu_1 \equiv \mu_0(2 \cdot z_0 \cdot \epsilon),$$

with $\mu_0(\cdot)$ as in Lemma 5.3, and with z_0 as in Lemma 6.2. Suppose that ϵ has been chosen so that $2 \cdot z_0 \cdot \epsilon$ and μ_1 are both less than 1/2. The next task is to evaluate $\mathfrak{s}_{\mu_1}([A'; h])$. To begin, it is useful to first compare the projections $\Pi(\cdot; A')$ and $\Pi(\cdot; A_0)$. For this purpose, the following observations prove useful:

Lemma 6.3. *Let M be a compact, oriented Riemannian 4-manifold and let G be a simple Lie group. There exists $\epsilon_0, r_0 > 0$ and for each nonnegative integer m, there exists $\xi_m < \infty$; and these numbers have the following significance: Let $P \to M$ be a principal G-bundle with characteristic classes (k, η) and let $[A] \in \mathfrak{b}_{\epsilon_0}(k, \eta) \equiv \mathfrak{b}'_{\epsilon_0}(k, \eta)/G$. Let $\omega \in L^2_3(P_-\Omega^2(\operatorname{Ad} P))$ be an L^2-normalized eigenvector for the operator $P_- d_A (P_- d_A)^*$ with eigenvalue $\mu(\omega)$. Then the following hold.*

(1) $\sup_{x \in M} |\omega|(x) < \xi_0 \cdot (1 + \mu)^2$.

(2) $\|\nabla_A \omega\|^2_{L^2} \le \xi_0 \cdot (1 + \mu)$.

(3) *If, in a ball of radius $r < r_0$ on M with center x, the connection A obeys assertion* (3) *of Lemma* 6.2, *then at points $y \in M$ with* $\operatorname{dist}(x, y) < r/4$, $|\nabla_A^{(m)} \omega|(y) \le \xi_m \cdot r^{-m} \cdot (1 + \mu)^2$.

Proof of Lemma 6.3. The first assertion is a consequence of Lemma 5.6, and the second follows using the Weitzenboch formula. The fourth is obtained via standard elliptic techniques (see [17, Chapter 6]).

Focus attention on an L^2-orthonormal basis of eigenvectors, $\{\omega_n : n \le N(8 \cdot \mu_1)\} \in L^2_3(P_-\Omega^2(\operatorname{Ad} P))$, for the operator $P_- d_{A_0} (P_- d_{A_0})^*$ with eigenvalues $\mu(\omega_n) < 8 \cdot \mu_1 < 1$. Here, μ_1 is defined in (6.1b). The bound $N(8 \cdot \mu_1)$ for the maximum size of such a basis depends only on G and the Riemannian metric (see Proposition A.1). Here, and elsewhere in this section, the convention will be to index the eigenvectors so that the eigenvalue of ω_n is not greater than that of ω_{n+1}. Introduce the notation from (4.2) and (4.10) and let

$$\omega'_n = \prod_\nu (1 - \beta_{\lambda(v)}) \cdot \omega_n. \tag{6.2}$$

Then $\{\omega'_n\}$ defines a set of L^2_3-sections of $P_-\Omega^2(\operatorname{Ad} P(w))$, and Lemma 6.3 implies that

$$|\langle \omega'_i, \omega'_n \rangle_{L^2} - \delta_{in}| \le z_1 \cdot \sum_\nu \lambda(v)^4 \le z_1 \cdot \epsilon^{3/2}/J, \tag{6.3}$$

$$|\|P_- d^*_{A'} \omega'_n\|^2_{L^2} - \mu(\omega_n)| \le z_1 \cdot \sum_\nu \lambda(v)^2 \le \epsilon^{3/4}. \tag{6.4}$$

Furthermore, any $\varphi \in L^2_1(P_-\Omega^2(\operatorname{Ad} P[w]))$ obeys

$$\begin{aligned} &|\langle P_- d^*_{A'} \varphi, P_- d^*_{A'} \omega'_n \rangle_{L^2} - \mu(\omega_n) \cdot \langle \varphi, \omega'_n \rangle_{L^2}| \\ &\quad \le z_1 \cdot (\|\nabla_{A'} \varphi\|_{L^2} + \|\varphi\|_{L^2}) \cdot \epsilon^{3/8}. \end{aligned} \tag{6.5}$$

It follows from (6.2)–(6.5) that there exists $\epsilon > 0$ which depends only on M and G and which is such that when $\epsilon < \epsilon_0$, then the assignment of $\{\omega_n : n \leq N(8 \cdot \mu_1)\}$ to $\{\Pi(8 \cdot \mu_1, A') \cdot \omega'_n\}$ defines a linear map from Range $\Pi(8 \cdot \mu_1; A_0) \subset L^2(P_-\Omega^2(\operatorname{Ad} P))$ into Range $\Pi(8 \cdot \mu_1; A') \subset L^2(P_-\Omega^2(\operatorname{Ad} P(w)))$ which is injective.

Conversely, if $\{\tau_n\} \in L^2_3(P_-\Omega^2(\operatorname{Ad} P(w)))$ is an L^2-orthonormal basis of eigenvectors for the operator $P_-d_{A'}(P_-d_{A'})^*$ with eigenvalues less than $2 \cdot \mu_1$, and if $\{\tau'_n\}$ is defined by (6.2), then $\{\tau'_n\}$ defines a set of L^2_3-section of $P_-\Omega^2(\operatorname{Ad} P)$ which, due to Lemma 6.3, obeys (6.3) and (6.4) with A_0 replacing A' in (6.4).

It follows from these last two observations that $\epsilon_0 > 0$ exists, which depends only on the Riemannian metric and on G and which is such that if $\epsilon < \epsilon_0$, then

(6.6)
$$\Pi(2 \cdot \mu_1; A') \cdot L^2(P_-\Omega^2(\operatorname{Ad} P(w))) \subset \operatorname{Span}\{\Pi(8 \cdot \mu_1; A') \cdot \omega'_n : n \leq N(4 \cdot \mu_1)\}.$$

Now, for $n \leq N(8 \cdot \mu_1)$, write each ω'_n as $\Pi(8 \cdot \mu_1; A') \cdot \omega'_n + \theta_n$ with $\theta_n \in (1 - \Pi(8 \cdot \mu_1; A')) \cdot L^2(P_-\Omega^2(\operatorname{Ad} P(w))) \cap L^2_3$ obeying

(6.7)
$$P_-d_{A'}(P_-d_{A'})^*\theta_n + (1 - \Pi(8 \cdot \mu_1; A')) \cdot \omega'_n = 0.$$

Because

$$\|P_-d^*_{A'}\theta_n\|^2_{L^2} = \langle P_-d^*_{A'}\theta_n, P_-d^*_{A'}\omega'_n\rangle_{L^2},$$

(6.5) implies that

(6.8)
$$\|\nabla_{A'}\theta_n\|^2_{L^2} + \|\theta_n\|^2_{L^2} \leq (1 + \mu_1^{-1}) \cdot z_2 \cdot \epsilon^{3/4} \leq z_2 \cdot \epsilon^{1/2}.$$

Here the facts that $[A', h] \in \mathfrak{b}'_\epsilon(P(w))$ and $\mu_1(\epsilon) \geq \epsilon^{1/4}$ have been used.

Observe: Equation (6.6) asserts that $\mathfrak{s}_{\mu_1}([A', h]) = 0$ if

(6.9)
$$\mathfrak{f}_n \equiv \langle \Pi(8 \cdot \mu_1; A') \cdot \omega'_n, P_-F_{A'} + P_-(d^*_{A'}v(\lambda_0; A') \wedge d^*_{A'}v(\lambda_0; A'))\rangle_{L^2} = 0$$

for all $n \leq N(4 \cdot \mu_1)$. Concerning $\mathfrak{f}_n$, one has (see (1.7) and Proposition 5.4 in [21])

Lemma 6.4. *There is a constant $c(G)$ which depends only on the group G, and there are constants $\epsilon_0 > 0$ and $z_0 < \infty$ which only depend on G and on M; these are such that when $\epsilon < \epsilon_0$, then for all $n \leq N(8 \cdot \mu_1)$,*

$$\left| \mathfrak{f}_n - \langle \omega_n, P_-F_{A_0}\rangle_{L^2} - \sum_{\nu} c(G)\lambda(v)^2 (\omega_n(x(v)), X(\varphi(v), f_0, [A_0, h_0], h_1(v)) \cdot P_+F_{A_1}(s)) \right| \leq z_0 \cdot \epsilon \cdot (J^{-1/16} \cdot R^2 + \epsilon^{1/4} \cdot R^4),$$

where $X(\cdot)$ is defined prior to Lemma 4.7.

Proof of Lemma 6.4. Due to (5.23),

$$|\mathfrak{f}_n - \langle \Pi(8\cdot\mu_1; A')\cdot\omega_n', P_-F_{A'}\rangle_{L^2}| \le z_0\cdot\epsilon^{3/2}. \tag{6.10}$$

To analyze the left-hand side, above, note that

$$\begin{aligned} P_-F_{A'} &= \prod_\nu (1-\beta_{\lambda(v)})\cdot P_-F_{A_0} \\ &\quad + \sum_\nu P_-\left(d\beta_{4\lambda(v)/t(\nu)}\wedge d^*_{A'}u(w(v),t(v)) + \beta_{4\lambda(v)/t(\nu)}\right. \\ &\quad \left.\cdot(1-\beta_{4\lambda(v)/t(\nu)})\cdot d^*_{A'}u(w(v),t(v))\wedge d^*_{A'}u(w(v),t(v))\right). \end{aligned} \tag{6.11}$$

Thus, due to Lemmas 4.8 and 5.9 and (6.1),

$$\begin{aligned} &\left|\mathfrak{f}_n - \langle \Pi(8\cdot\mu_1;A')\cdot\omega_n', \prod_\nu(1-\beta_{\lambda(v)}\cdot P_-F_{A_0}\right. \\ &\qquad \left. + \sum_\nu d\beta_{4\lambda(v)/t(\nu)}\wedge d^*_{A'}u(w(v),t(v))\rangle_{L^\epsilon}\right| \le z_0\epsilon^{3/2}. \end{aligned} \tag{6.12}$$

The third term on the left-hand side of (6.12) is evaluated via an integration by parts:

$$\begin{aligned} &|\langle\Pi(8\cdot\mu_1;A')\cdot\omega_n', d\beta_{4\lambda(v)/t(\nu)}\wedge d^*_{A'}u(w(v),t(v))\rangle_{L^2} \\ &\qquad - \langle\omega_n', \beta_{4\lambda(v)/t(\nu)}\cdot P_-d_{A'}d^*_{A'}u(w(v),t(v))\rangle_{L^2}| \\ &\quad \le z_0\cdot(r\cdot\|P_-d^*_{A'}\Pi(8\cdot\mu_1;A')\cdot\omega_n'\|_{L^2;B[v]} \\ &\qquad + \|\theta_n\|_{L^2;B[v]})\cdot R^2\cdot\epsilon/\sqrt{N}. \end{aligned} \tag{6.13}$$

Here $B[v]\equiv B(x(v),4\cdot\lambda(v)/t(v))$.

Equation (6.13) is readily evaluated using (4.15). The evaluation results in

$$\begin{aligned} &|\langle\Pi(8\cdot\mu_1;A')\cdot\omega_n', d\beta_{4\lambda(v)/t(\nu)}\wedge d^*_{A'}u(w(v),t(v))\rangle_{L^2} \\ &\qquad - c(G)\cdot\lambda(v)^2\cdot(\omega_n(x(v)), X(f(v),h_0(v),h_1(v))\cdot P_+F_{A_1}(s))| \\ &\quad \le z_0\cdot(\|\nabla_{A_0}\omega_n\|^{1/4}_{L^2;B[v]}\cdot\lambda(v)^{7/4}\cdot(\lambda(v)/r)^{1/4} + r\cdot R^4\cdot\epsilon^2/J \\ &\qquad + (r\cdot\|P_-d^*_{A'}\Pi(8\cdot\mu_1;A')\cdot\omega_n'\|_{L^2;B[v]} + \|\theta_n\|_{L^2;B[v]})\cdot R^2\cdot\epsilon/\sqrt{J}). \end{aligned}$$

This last estimate utilizes assertion (3) of Lemma 6.3.

By substituting this last equation into (6.12), and using (6.8) and Lemma 6.3, one obtains the final estimate.

Part 2: The choice of gluing sites. The proof proper of Proposition 6.1 begins here with the start of the construction of a family of homotopy equivalences between a given $\mathfrak{B}'(k,\eta)$ and $\mathfrak{B}'(k+c(G)\cdot J,\eta)$. These homotopy equivalences will require certain parameter specifications. The specification of these free parameters will constitute the last parts of this section, and complete the proof of the proposition.

For $\epsilon_0 > 0$, but yet to be specified, and for $\epsilon \in (0, \epsilon_0)$, suppose that $U \subset \mathfrak{b}'_\epsilon(k, \eta)$ has been specified as in the statement of Proposition 6.1. The proof begins with five lemmas which detail the choice of points in M about which to make the gluing construction of (4.2) and (4.10). Choosing this set of points constitutes this part.

To start, choose r_1 to be much less than the injectivity radius of M. A further restriction on r_1 comes from

Lemma 6.5. *Let M be a compact, oriented, 4-dimensional Riemannian manifold, and let G be a simple Lie group. There exists $q > 0$ with the following significance: Let (k, η) be the characteristic classes for a principal G-bundle $P \to M$. Fix $\mu > 0$ and $\epsilon > 0$. Let $W \subset \mathfrak{B}(k, \eta)$ be a compact set. For each $[A] \in U \equiv \mathfrak{M}(k, \eta) \cup W$, let $\Pi(\mu; A)$ denote the L^2-orthogonal projection in $L^2(P_-\Omega^2(\operatorname{Ad} P))$ onto the span of the set of eigenvectors of the operator $(P_- d_A)d_A^*$ with eigenvalue less than μ. There exists $r_0 > 0$ so that for any $r \in (0, r_0]$, there is a set of disjoint balls $\{B(x[j], r)\} \subset M$ with the property that for each $[A] \in U$, a subset $\{B(x[j(A)], r)\} \subset \{B(x[j], r)\}$ obeys*

(1) *For each index $j(A)$, $[A] \in \mathfrak{B}_{\epsilon/2}(P; x[j(A)], r)$.*

(2) *For each $\omega \in \operatorname{Range} \Pi(\mu; A)$,*

$$\sum_{j(A)} \|\omega\|^2_{L^2; B(x[j(A)], r/4)} \geq q \cdot \|\omega\|^2_{L^2}.$$

Prior to proving this lemma, it is useful to make a preliminary observation:

Lemma 6.6. *Let M be a compact, oriented, 4-dimensional Riemannian manifold, and let G be a simple Lie group. Let (k, η) be the characteristic classes for a principal G-bundle $P \to M$ for which $\mathfrak{M}(k, \eta)$ is nonempty. Fix $\mu > 0$. The set $\Xi(k) \equiv \bigcup_{0 \leq k' \leq k} \{|\omega| : \omega \in \operatorname{Range} \Pi(\mu; A), \|\omega\|_{L^2} = 1$ and $[A] \in \mathfrak{M}(k', \eta)\}$ is a compact set in $L^2_1(M)$.*

Proof of Lemma 6.6. Proposition A.1 in the appendix establishes a metric dependent, but k independent, upper bound for the dimension of the range of $\Pi(\mu; A)$ for $[A] \in \mathfrak{M}(k, \eta)$. Knowledge of the noncompactness of $\mathfrak{M}(k, \eta)$ from [9] (see also §5 of [22]) details the behavior of a sequence $[A(i)] \in \mathfrak{M}(k, \eta)$ which fails to converge. With this knowledge, plus Lemma 6.3, a straightforward cut and paste mimicking (6.2)–(6.5) establishes the lemma.

Proof of Lemma 6.5. Note first that the compactness of W plus Proposition A.1 insures that $\dim(\operatorname{Range} \Pi(\mu; A))$ is a priori bounded for $[A] \in U$. Then, given Lemma 6.6, it follows that $\{|\omega| : \omega \in \operatorname{Range} \Pi(\mu; A), \|\omega\|_{L^2} = 1$ and $[A] \in W\}$ is a compact set in $L^2_1(M)$. Exploit this fact by copying the proof of Lemma 4.5 to establish the existence of constant $q > 0$ with the following properties: First, q depends only on the group G and the Riemannian metric. Second, given $r > 0$, but sufficiently small, there exists a set of disjoint balls

$\{B(x[j], r)\}$ such that for any $[A] \in U$ and $\omega \in \operatorname{Range} \Pi(\mu; A)$,

$$\sum_j \|\omega\|^2_{L^2; B(x[j], r/4)} \geq 2 \cdot q \cdot \|\omega\|^2_{L^2}.$$

Since W is compact, given $\epsilon > 0$, there exists $r_1 > 0$ such that $W \subset \mathfrak{B}_{\epsilon/2}(P; x[j], r)$ whenever $r < r_1$. For $[A] \in \mathfrak{M}(k, \eta)$, the condition $[A] \notin \mathfrak{B}_{\epsilon/2}(P; x[j], r)$ can be true for at most $2 \cdot k/\epsilon$ of the indices j. For sufficiently small r, Lemma 6.3 insures that the deletion of $2 \cdot k/\epsilon$ balls from the sum above will not effect the validity of the inequality if $2 \cdot q$ is replaced by q on the right-hand side. This completes the proof of Lemma 6.5.

Let $W \subset \mathfrak{B}'(k, \eta)$ be a compact set as given in the statement of Proposition 6.1. Consider $U \equiv \mathfrak{M}'(k, \eta) \cup W$. Choose $r_1 > 0$, but sufficiently small so that Lemma 6.5 can be invoked for the projection of U into $\mathfrak{B}(k, \eta)$ using $\mu \equiv 16\mu_0(2 \cdot z_0 \cdot \epsilon)$, with $\mu_0(\cdot)$ as in Lemma 5.3, and with z_0 as in Lemma 6.2. By adding balls if necessary, one can assume that the number of points $\{x[j]\} \subset M$ is equal to $J_1 \equiv c_0/r_1^4$, with c_0 fixed by the Riemannian metric on M.

Construct a homotopy $\Phi \equiv \Phi(\{x[j], r_1\})\colon [0,1] \times \mathfrak{B}'(k, \eta) \to \mathfrak{B}'(k, \eta)$ of Lemma 6.2. Then $U_0 \equiv \Phi(1, U) \subset \mathfrak{b}'_{z_0 \cdot \epsilon}(k, \eta)$ with $z_0 < \infty$ as provided by G and M via Lemma 6.2. This homotopy Φ can be constructed for any sufficiently small, but positive, choice of r_1 and balls $\{B(x[j], r_1)\}$ which satisfy the conclusions of Lemma 6.5. The next lemma gives an additional upper bound on the number r_1.

Lemma 6.7. *Let M be a compact, oriented Riemannian 4-manifold and let G be a simple Lie group. There exist $\epsilon_0 \in (0, 1]$ and $c_0 < \infty$ with the following significance: Let $P \to M$ be a principal G-bundle with characteristic classes (k, η), and for $\epsilon \in (0, \epsilon_0]$, let $W \subset \mathfrak{b}'_\epsilon(k, \eta)$ be a given, compact set. Set $U \equiv \mathfrak{M}'(k, \eta) \cup W$. Fix $\mu_1 \equiv \mu_0(2 \cdot z_0 \cdot \epsilon)$ with $\mu_0(\cdot)$ as in Lemma 5.3 and z_0 as in Lemma 6.2. Let q be as in Lemma 6.5. Then $r_0(\epsilon) > 0$ exists such that when $r_1 \in (0, r_0)$, there is a set of $J_1 \equiv c_0/r_1^4$ balls $\{B(x[j], r_1) \subset M\}$ which satisfy*

(1) *The conclusion of Lemma 6.5.*

(2) *Let $U_0 \equiv \Phi(1, U)$, with $\Phi \equiv \Phi(\{x[j], r_1\})$ as described in Lemma 6.2. For each $[A, h] \in U_0$, and for each $\omega \in \operatorname{Range} \Pi(8 \cdot \mu_0; A)$,*

$$\sum_j \|\omega\|^2_{L^2; B(x[j], r_1/4)} > q/2 \cdot \|\omega\|^2_{L^2}.$$

Proof of Lemma 6.7. This is a straightforward perturbation argument using assertion (1) of Lemma 6.2 and Proposition A.1. The details are left to the reader.

Given $U \equiv \mathfrak{M}'(k,\eta) \cup W$, and having constrained r_1 by Lemma 6.7, let $\{B[j] \equiv B(x[j]; r_1/4)\}$ be the set of balls which said lemma provides. Assume that a base point $x_0 \in M$ has been chosen which is disjoint from each $B[j]$. For each j, choose a smooth path $\varphi[j]$ from x_0 to $x[j]$. Then, for each $x \in B[j]$, define a path $\varphi(x)$ from x_0 to x by first following $\varphi[j]$ to $x[j]$, and then by following the radial geodesic from $x[j]$ to x. Note that the chosen frame f_0 in $\operatorname{Fr} M|_{x_0}$ defines, by parallel transport with the Riemannian connection to $x[j]$ and then to x, a frame f for $\operatorname{Fr} M$ over each ball $B[j]$.

It is necessary to subdivide each ball $B[j]$ two times. For this purpose, fix $r_2 \in (0, r_1/8]$. The next lemma defines the first subdivision.

Lemma 6.8. *The Riemannian metric fixes $q_1, c_1 > 0$ and $\epsilon_0 \in (0,1]$ with the following significance: With $\epsilon \in (0,\epsilon_0]$, let the set U_0, the set of balls $\{B(x[j], r_1)\}$ and $\mu_1 > 0$ be given in Lemma 6.7. There exists $r_1' < r_1/8$ and for each $r_2 \in (0, r_1')$ and for each j, there exist J_2 ($\equiv$ the closest integer to $c_1 \cdot r_1^4/r_2^4$) disjoint balls $\{B[j,k] \equiv B(x[j,k], r_2)\}$ in $B[j]$ with the property that when $[A_0, h_0] \in U_0$, and when $\omega \in \operatorname{Range}\Pi(8\cdot\mu_1; A_0)$, then*

$$\sum_j \sum_k \|\omega\|^2_{L^2; B(x[j,k], r_2)} > q_1 \cdot \|\omega\|^2_{L^2}.$$

In addition, there exist at least $q_1 \cdot J_1 \cdot J_2$ of the $J_1 \cdot J_2$ indices in the set $\{(j,k)\}$ with the property that $\inf_{x \in B[j,k]} |\omega|(x) > q_1 \cdot \|\omega\|_{L^2}$.

Proof of Lemma 6.8. The proof of the first assertion is obtained by mimicking, again, the proof of Lemma 4.5; the argument is straightforward, and again, it is omitted. The second assertion follows from the first assertion with Lemma 6.3.

Lemma 6.8 has the following useful corollary:

Lemma 6.9. *The Riemannian metric provides $q_0, \epsilon_0 \in (0,1]$ with the following significance: Assume that $\epsilon < \epsilon_0$. Let μ_1 be as in Lemma 6.7, and let $N(8\cdot\mu_1)$, as provided by Proposition A.1, be such that for any $[A] \in \mathfrak{b}_{\epsilon_0}(k,\eta)$, $N(8\cdot\mu_1)$ bounds the dimension of $\operatorname{Range}\Pi(8\cdot\mu_1; A)$. Let U_0 be as in the preceding lemma and Lemma 6.7. Then, for each $[A_0, h_0] \in U_0$, there are $q_0 \cdot J_1 \cdot J_2$ disjoint subsets $\{\Lambda(\alpha) \subset \{(j,k)\}\}$ which have the three properties listed below. First, no $\Lambda(\alpha)$ contains more than $N(8\cdot\mu_1)$ elements. Second, for each α, the assignment of $\omega \in \operatorname{Range}\Pi(8\cdot\mu_1; A_0)$ to $I(\Lambda(\alpha))\cdot\omega \equiv (\omega(x[j,k]))_{(j,k]\in\Lambda[\alpha)} \in \bigoplus_{(j,k]\in\Lambda[\alpha)} P_-\Omega^2(\operatorname{Ad} P)|_{x[j,k]}$ defines a linear injection. Third, if $|I(\Lambda(\alpha))\cdot\omega|^2$ is defined to be equal to $\sum_{(j,k]\in\Lambda[\alpha)} |\omega(x[j,k])|^2$, then*

$$\inf_{0\neq\omega\in\operatorname{Range}\Pi(8\cdot\mu_1;A_0)} (|I(\Lambda(\alpha))\cdot\omega|^2/\|\omega\|^2_{L^2}) > q_0.$$

Proof of Lemma 6.9. The proof is by induction on the integers in the set $\{1, \cdots, N(8 \cdot \mu_1)\}$; these are the allowable dimensions of the nontrivial subspaces of $\operatorname{Range}\Pi(8 \cdot \mu_1; A_0)$. For integer $p = 1$, pick a normalized $\omega(1) \in \operatorname{Range}\Pi(8 \cdot \mu_1; A_0)$. Then ω defines a 1-dimensional subspace $L(1)$ of $\operatorname{Range}\Pi(8 \cdot \mu_1; A_0)$. Lemma 6.8 asserts that there are at least $q_1 \cdot J_1 \cdot J_2$ indices $\{\Lambda(\alpha)\} \in \{(j,k)\}$ with the property that the assignment of $\omega \in L(1)$ to $I(\Lambda(\alpha)) \cdot \omega \equiv (\omega(x[j,k]))_{(j,k)\in\Lambda(\alpha)} \in \bigoplus_{(j,k)\in\Lambda(\alpha)} P_-\Omega^2(\operatorname{Ad} P)|_{x[j,k]}$ defines a linear injection with

$$\inf_{0\neq\omega\in L[1]}(|I(\Lambda(\alpha)) \cdot \omega|^2/\|\omega\|^2_{L^2}) > q_1.$$

To construct the induction step from integer p to integer $p+1$, it is necessary to introduce the constant z_0 which, according to Lemma 6.3, is provided by the Riemannian metric, and which has the property that when $\omega \in \operatorname{Range}\Pi(8 \cdot \mu_1)$, then $\|\omega\|_\infty \leq z_0 \cdot \|\omega\|_{L^2}$.

Now, suppose that for $p \geq 1$ there exists $q_p \in (0, \max(q_1, z_0^2/2))$, and a p-dimensional subspace, $L(p) \subset \operatorname{Range}\Pi(8 \cdot \mu_1; A_0)$ with the property that $q_p \cdot J_1 \cdot J_2$ disjoint subsets $\{\Lambda(\alpha) \subset \{(j,k)\}; \alpha = 1, \cdots, q_p \cdot J_1 \cdot J_2\}$ exist for which the assignment of $\omega \in L(p)$ to $I(\Lambda(\alpha)) \cdot \omega \equiv (\omega(x[j,k]))_{(j,k)\in\Lambda(\alpha)} \in \bigoplus_{(j,k)\in\Lambda(\alpha)} P_-\Omega^2(\operatorname{Ad} P)|_{x[j,k]}$ defines a linear injection with

$$\inf_{0\neq\omega\in L(p)}(|I(\Lambda(\alpha)) \cdot \omega|^2/\|\omega\|^2_{L^2}) > q_p.$$

Assume further that each $\Lambda(\alpha)$ contains at most p elements of the indexing set $\{(j,k)\}$.

Let $\omega(p+1) \in \operatorname{Range}\Pi(8 \cdot \mu_1; A_0)$ be an L^2-normalized element in the L^2-orthogonal complement to $L(p)$. Let $L(p+1)$ denote the linear span of $\{\omega(p+1), L(p)\}$. Suppose that there exists at least $q_p \cdot J_1 \cdot J_2/(2 \cdot p)$ of the set of subsets $\{\Lambda(\alpha)\}$ for which the map $I(\Lambda(\alpha))$ on $L(p+1)$ obeys

$$\inf_{0\neq\omega\in L(p)}(|I(\Lambda(\alpha)) \cdot \omega|^2/\|\omega\|^2_{L^2}) \leq q_p/2.$$

Relabel so that the preceding equation is true for the subsets $\{\Lambda(\alpha) \colon \alpha \leq q_p \cdot J_1 \cdot J_2/(2 \cdot p)\}$.

For each $\alpha \leq q_p \cdot J_1 \cdot J_2/(2 \cdot p)$, let $L'(\alpha) \subset L(p+1)$ denote the maximal subspace of $L(p+1)$ with the property that for the subsets $\Lambda(\alpha)$, the map $I(\alpha)$ obeys

$$\sup_{0\neq\omega\in L'}(|I(\Lambda(\alpha)) \cdot \omega|^2/\|\omega\|^2_{L^2}) \leq q_p/2.$$

This subspace has dimension at most 1. If it had dimension 2 or more, there would be a nontrivial kernel for the projection of $L'(\alpha)$ onto $\operatorname{Span}\{\omega(p+1)\}$ and the preceding equation could not be obeyed on that kernel. Let $\omega'(\alpha)$

generate $L'(\alpha)$. There are at least $q_1 \cdot J_1 \cdot J_2$ indices in the set $\{(j,k)\}$ with the property that $|\omega'(\alpha)|(x[j,k]) > q_1 \cdot \|\omega'(\alpha)\|_{L^2}$.

Since the total number of indices in $\bigcup\{\Lambda(\alpha) : \alpha \le q_p \cdot J_1 \cdot J_2/(2\cdot p)\}$ is at most $q_1 \cdot J_1 \cdot J_2/2$, it is possible to choose a set of indices $K \equiv \{(j(\alpha), k(\alpha)) : \alpha \le q_p \cdot J_1 \cdot J_2/(2 \cdot p)\} \subset \{(j,k)\}$ with the following two properties: First, K is disjoint from $\bigcup\{\Lambda(\alpha) : \alpha \le q_p \cdot J_1 \cdot J_2/(2 \cdot p)\}$. Second, for each α, $|\omega'(\alpha)|(x[j(\alpha), k(\alpha)]) > q_1 \cdot \|\omega'(\alpha)\|_{L^2}$.

Set $q_{p+1} \equiv \min(q_p/(2 \cdot p), q_p^2/(4 \cdot z_0^2))$ and for $\alpha \le q_{p+1} \cdot J_1 \cdot J_2$, define $\Lambda'(\alpha) \equiv \Lambda(\alpha) \cup (j(\alpha), k(\alpha))$. The sets $\{\Lambda'(\alpha)\}$ are mutually disjoint; and none contain more than $p+1$ indices. Finally, for each $\alpha \le q_{p+1} \cdot J_1 \cdot J_2$,

$$\inf_{0 \ne \omega \in L(p+1)} (|I(\Lambda'(\alpha)) \cdot \omega|^2 / \|\omega\|_{L^2}^2) > q_{p+1}.$$

This last fact follows from Lemma 6.3 with some simple algebra. The preceding equation completes the induction step from p to $p+1$.

Given the set $U \equiv \mathfrak{M}'(k,\eta) \cup W$, and the homotoped set U_0 from Lemma 6.7, Lemmas 6.7 and 6.8 construct a set of disjoint balls $\{B[j,k]\}$. It is necessary to further subdivide each ball $B[j,k]$. For this purpose, fix $r_3 \in (0, r_2/8]$, and fix $R \in (4/(c_0 \cdot c_1^2), \epsilon^{-1/4}]$; these constants are defined in Lemmas 6.7 and 6.8. No generality is lost in requiring c_1 from Lemma 6.8 to be such that for each (j,k), there exist J_3 ($\equiv$ the closest integer to $c_1 \cdot r_2^4/r_3^4$) disjoint balls $\{B[j,k,i] \equiv B(x[j,k,i], r_3)\}$ in $B[j,k]$; the center of $B[j,k,i]$ is the point $x[j,k,i]$. The value of r_3 is determined by

Lemma 6.10. *Let G be a simple Lie group. There exists an integer $m < \infty$ with the following significance: Let $\{\sigma_i\}_{i=1}^3$ be an orthonormal basis for $\mathfrak{su}(2)$; and let $\rho\colon \mathrm{SU}(2) \to G$ be a homomorphism which generates $\pi_3(G)$. Set $\{\tau_i \equiv \rho_*\sigma_i\}$. There are m points $\{g_\gamma\} \subset G$ such that the map $\Upsilon\colon (0,\infty)^m \to \mathfrak{g} \otimes \mathfrak{su}(2)$ which sends (s_γ) to*

$$\Upsilon(s_\gamma) \equiv \sum_{\gamma=1}^{m} s_\gamma \cdot \sum_{i=1}^{3} (g_\gamma \cdot \tau_i \cdot g_\gamma^{-1}) \otimes \sigma_i$$

is a surjection with the property that if $K \subset \mathfrak{g} \otimes \mathfrak{su}(2)$ is a compact set, there exists a compact set $K' \subset (0,\infty)^m$ which is mapped surjectively onto K by Υ.

Proof of Lemma 6.10. Observe first that by defining $(u_j \equiv \exp(\pi \cdot \sigma_j/2))_{j=1}^3 \in \mathrm{SU}(2)$ we have

$$\sum_{j=1}^{3} \sum_{i=1}^{3} (\rho(u_j) \cdot \tau_i \cdot \rho(u_j)^{-1}) \otimes \sigma_i = -\sum_{i=1}^{3} \tau_i \otimes \sigma_i.$$

Therefore, at the expense of replacing the integer m in the statement of the lemma by $4 \cdot m$, no generality is lost by considering such a map Υ as a map from $\mathbf{R}^m \to \mathfrak{g} \otimes \mathfrak{su}(2)$. The assertion that $m < \infty$ and $\{g_\gamma\}_{\gamma=1}^m \subset G$ exist such that $\Upsilon\colon \mathbf{R}^m \to \mathfrak{g} \otimes \mathfrak{su}(2)$ is surjective is equivalent to the assertion that $m < \infty$ and $\{g_\gamma\}_{\gamma=1}^m \subset G$ exist such that

$$\operatorname{Span}\left\{\sum_{i=1}^{3}(g_\gamma \cdot \tau_i \cdot g_\gamma^{-1}) \otimes \sigma_i\right\} = \mathfrak{g} \otimes \mathfrak{su}(2).$$

This last assertion follows from Lemma 4.10. Furthermore, as a map from $\mathbf{R}^m$, Υ is linear; this implies that a compact set in $\mathfrak{g} \otimes \mathfrak{su}(2)$ is covered by a compact set in $(0, \infty)^m$.

To return to the proof of Proposition 6.1, set the ratio r_2/r_3 so that $J_3 \equiv m$ of Lemma 6.10.

Part 3: Defining the parameters. Let $\epsilon_0 > 0$ be specified according to Lemmas 6.2–6.9 and fix $\epsilon \in (0, \epsilon_0]$. Let $W \subset \mathfrak{b}'_\epsilon(k, \eta)$ be a given compact set, let $U \equiv \mathfrak{M}'(k, \eta) \cup W$, and construct $U_0 \equiv \Phi(1, U)$ as detailed in Part 2 of this section, Lemma 6.7. Suppose that a smooth map $\varepsilon\colon \mathfrak{B}'(k, \eta) \to (0, \epsilon]$ has been defined; require that $\varepsilon(\cdot) \geq \min(\epsilon, \mathfrak{a}(\cdot))$, but leave ε otherwise unrestricted for now. For each index (j, k, i), suppose also that a smooth map $s[j, k, i]\colon \mathfrak{B}'(k, \eta) \to (0, 1]$ has been specified. Set

$$\lambda^2[j, k, i] \equiv s[j, k, i] \cdot R \cdot \varepsilon / J, \tag{6.14a}$$

With $J \equiv J_1 \cdot J_2 \cdot J_3$. Set

$$t[j, k, i] \equiv 16 \cdot \lambda[j, k, i]/r_3. \tag{6.14b}$$

Note that with $v \equiv (j, k, i)$ and $r \equiv r_3$, (6.1) is obeyed. For each index (j, k), set $h_1[j, k, i] \in P_{G_+}|_s$ according to the rule $h_1[j, k, i = \gamma] \equiv h_1 \cdot g_\gamma$ with $h_1 \in P_{G_+}|_s$ fixed and with $\{g_\gamma\}$ as in Lemma 6.9.

For fixed $v \equiv (j, k, i)$, and for fixed $[A_0, h_0] \in \mathfrak{B}'(k, \eta)$, the data in the preceding paragraph provides all required data for the definition of $w([A_0, h_0])(v) \equiv [[A_0, h_0], f_0, f[v], \varphi[v], \lambda[v]/t[v], t[v], [A_1, h_1[v]]]$, and then for the construction of the orbit $[A'(w([A_0, h_0])(v), h_0] \in \mathfrak{B}'(k + c(G), \eta)$. By making this construction simultaneously over the index set $\{v \equiv (j, k, i)\}$, one defines a smooth map

$$T\colon \mathfrak{B}'(k, \eta) \to \mathfrak{B}'(k + c(G) \cdot J, \eta), \tag{6.15}$$

which is a homotopy equivalence (see Proposition 4.1).

Part 4: From local to global. To complete the proof of Proposition 6.1, it remains to make some specific choices for the data $(\epsilon_0, r_1, r_2, R, \varepsilon, \{s(j,k,i)\})$. The following digression is required for the purpose of setting up the formalism with which the ultimate choice of the data will be made. The formalism below is meant to make a global version over $U \equiv \mathfrak{M}'(k,\eta) \cup W$ of the local construction of $\Lambda(\alpha)[\cdot]$ and $l[\alpha]$ of Lemma 6.11, below.

Given the set $U \equiv \mathfrak{M}'(k,\eta) \cup W$, and the homotoped set U_0 from Lemma 6.7, fix $[A_0, h_0] \in U_0$. Let $\{\Lambda(\alpha) \equiv \Lambda(\alpha)[A_0, h_0] \subset \{(j,k)\} : \alpha \le q_0 \cdot J_1 \cdot J_2\}$ be as provided for $[A_0, h_0]$ by Lemma 6.9 and let $|\Lambda(\alpha)|$ denote the size of the set $\Lambda(\alpha)$. Let

$$V(\alpha) \equiv V(\alpha)[A_0, h_0] \equiv \bigoplus_{(j,k)\in\Lambda(\alpha)[A_0,h_0]} P_-\Omega^2(\operatorname{Ad} P)|_{x[j,k]}.$$

Define a metric on $V(\alpha)$ by sending $\varphi_{1,2} \equiv (\varphi_{1,2}(j,k))_{(j,k)\in\Lambda(\alpha)}$ to $\langle\varphi_1, \varphi_2\rangle_{V(\alpha)} \equiv |\Lambda(\alpha)|^{-1} \cdot \sum_{(j,k)} (\varphi_1[j,k], \varphi_2[j,k])$. With this metric on $V(\alpha)$, and with the L^2-metric on $\operatorname{Range}\Pi(8\cdot\mu_1; A_0)$, the adjoint $I(\Lambda(\alpha))^* : V(\alpha) \to \operatorname{Range}\Pi(8\cdot\mu_1; A_0)$ is well defined.

The choice of path $\varphi[j,k,i]$ from the base point to $x[j,k,i]$ defines an isometric identification of each $V(\alpha)$ with the vector space $\bigoplus_{|\Lambda(\alpha)|}(\mathfrak{g}\otimes\mathfrak{su}(2))$ (by parallel transport of f_0 with the Levi-Civita connection and of h_0 with A_0). With this identification, Lemma 6.10 provides a surjection, $\Upsilon(\alpha) : (0,\infty)^{k(\alpha)} \to V(\alpha)$, where $k(\alpha) \equiv k(\alpha)[A_0, h_0] \equiv m \cdot |\Lambda(\alpha)|$.

Lemma 6.11. *The Riemannian metric provides constants $z_0 \in [8,\infty)$, $z_1, e_0 > 0$, and $\epsilon_0 \in (0,1]$ with the following significance: For $\epsilon \in (0,\epsilon_0]$, let U_0 and μ_1 be as described in Lemma 6.7. Let $\{(j,k)\}$ be the set of indices which are provided by Lemma 6.8. Let q_0, and, given $[A_0, h_0] \in U_0$, $\{\Lambda(\alpha) \subset \{(j,k)\} : \alpha \le q_0 \cdot J_1 \cdot J_2\}$ be as provided by Lemma 6.9. For each α, let $k(\alpha) \equiv m \cdot |\Lambda(\alpha)|$ with m as in Lemma 6.10. Define a Riemannian metric on $(0,\infty)^{k(\alpha)}$ by taking the product of the Euclidean metrics on $(0,\infty)$. Let $l(\alpha) \equiv I(\Lambda(\alpha))^* \circ \Upsilon(\alpha) : (0,\infty)^{k(\alpha)} \to \operatorname{Range}\Pi(8\cdot\mu_1; A_0)$. The restriction of $l(\alpha)$ to $(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\alpha)}$ defines a map from $(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\alpha)}$ into $\{\omega \in \operatorname{Range}\Pi(8\cdot\mu_1; A_0) : \|\omega\|_{L^2} < \epsilon_0\}$ which is a surjection of both $(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\alpha)}$ and $(4\cdot z_0^{-1}\cdot\epsilon_0, \frac{1}{4}\cdot z_0\cdot\epsilon_0)^{k(\alpha)}$ onto an open neighborhood of $0 \in \operatorname{Range}\Pi(8\cdot\mu_1; A_0)$ which contains $\{\omega \in \operatorname{Range}\Pi(8\cdot\mu_1; A_0) : \|\omega\|_{L^2} < z_1\cdot\epsilon_0\}$. The differential $dl[\alpha]$ has adjoint $dl[\alpha]^* : \mathbf{R}^{\dim(\operatorname{Range}\Pi(8\cdot\mu_1; A_0))} \to T(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\alpha)}$ which obeys*

$$\inf_{0\neq\omega}\left(|dl[\alpha]^*\cdot\omega| / \|\omega\|_{L^2}\right) > e_0.$$

Proof of Lemma 6.11. This is a direct consequence of Lemmas 6.9 and 6.10.

To apply Lemma 6.11, use it to define an open set in U_0 in the following way: Given $[A_0, h_0] \in U_0$, there exists an open set $X \equiv X[A_0, h_0] \subset U_0$ and an open set $X' \in \mathfrak{B}'(k, \eta)$ which restricts to U_0 as X, and a number $\mu[A_0, h_0] \in (7 \cdot \mu_1, 8 \cdot \mu_1)$ such that the assignment of $[A, h] \in X'$ to the composition of the map $\Upsilon(\alpha)[A_0, h_0]\colon (z_0^{-1} \cdot \epsilon_0, z_0 \in \epsilon_0)^{k(\alpha)[A_0,h_0]} \to V(\alpha)[A_0, h_0]$, with the adjoint of the restriction map $I(\Lambda(\alpha)[A_0, h_0])^*\colon V(\alpha)[A_0, h_0] \to \operatorname{Range}\Pi(\mu[A_0, h_0]; A)$, defines a smooth map $Y(\alpha)_{[A_0,h_0]}\colon X \times (z_0^{-1} \cdot \epsilon_0, z_0 \cdot \epsilon_0)^{k(\alpha)[A_0,h_0]} \to L^2(P\_\Omega^2(\operatorname{Ad} P))$.

The sets X and X' can be chosen to have special properties: (1) Require that X' be smoothly contractible. (2) Require that the assignment of $[A, h] \in X'$ to the L^2-orthogonal projection $\Pi(\mu[A_0, h_0]; A)$ define a smoothly varying family of projections on $L^2(P\_\Omega^2(\operatorname{Ad} P))$. (3) Require that the restricted map $Y(\alpha)_{[A_0,h_0]}([A, h], \cdot)\colon (z_0^{-1} \cdot \epsilon_0, z_0 \cdot \epsilon_0)^{k(\alpha)[A_0,h_0]} \to \operatorname{Range}\Pi(\mu[A_0, h_0]; A)$ is a surjection whose differential has adjoint which obeys

$$\inf_{0 \neq \omega \in \operatorname{Range}\Pi(8 \cdot \mu_1; A)} (|dY(\alpha)_{[A_0,h_0]}([A, h], \cdot)^* \cdot \omega| / \|\omega\|_{L^2}) > e_0. \tag{6.16}$$

The last required property is described by

Lemma 6.12. *Let ϵ_0 and z_0 be as in Lemma 6.11. The sets X and X', above, can be chosen to have the following property: There exists a smooth map $\varsigma_0(\alpha)\colon X' \to (4 \cdot z_0^{-1} \cdot \epsilon_0, 1/4 \cdot z_0 \cdot \epsilon_0)^{k(\alpha)[A_0,h_0]}$ with the property that*

$$Y(\alpha)_{[A_0,h_0]}([A, h], \varsigma_0(\alpha)([A, h])) = 0 \quad \textit{for any } [A, h] \in X'.$$

Proof of Lemma 6.12. This is a direct consequence of Lemma 6.11 and the inverse function theorem.

Note that the assignment of the vector space $\operatorname{Range}\Pi(\mu[A_0, h_0]; A)$ to a pair (A, h) with $[A, h] \in X'$ defines a smooth vector bundle over X' which is isomorphic to a product vector bundle $X' \times \mathbf{R}^{n[A_0,h_0]}$; with

$$n[A_0, h_0] \equiv \dim(\operatorname{Range}\Pi(\mu[A_0, h_0]; A)\colon [A] \in X').$$

This isomorphism is obtained by making a $\mathfrak{G}(P)$-equivariant choice of L^2-orthonormal basis for $\operatorname{Range}\Pi(\mu[A_0, h_0]; A)$ as (A, h) vary through $\pi^{-1}(X')$, where $\pi\colon \mathfrak{A}(P) \times P|_{x_0} \to \mathfrak{B}'(k, \eta)$ is the defining projection.

Let $\{\omega_n(\cdot)\}_{n \leq n[A_0,h_0]}$ denote such a choice. Having made the choice, each map $Y(\alpha)_{[A_0,h_0]}$ $(\alpha \leq q_0 \cdot J_1 \cdot J_2)$ is defined as

$$Y(\alpha)_{[A_0,h_0]}\colon X' \times (z_0^{-1} \cdot \epsilon_0, z_0 \cdot \epsilon_0)^{k(\alpha)[A_0,h_0]} \to \mathbf{R}^{n[A_0,h_0]}. \tag{6.17}$$

Pick an open set $Z' \subset X'$ with compact closure in X'. Require that Z' restricts to X as an open set Z with compact closure in X. Since each X' is contractible, each map $Y(\alpha)_{[A_0,h_0]}$ can be deformed to a smooth map,

$$Y'(\alpha)_{[A_0,h_0]}\colon X' \times (z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\alpha)[A_0,h_0]} \to \mathbf{R}^{n[A_0,h_0]} \tag{6.18}$$

which agrees with $Y(\alpha)_{[A_0,h_0]}$ on Z'; and which has the following special properties.

Lemma 6.13. *Let ϵ_0, z_0, z_1 and e_0 be as in Lemma* 6.11. *The sets Z' and X', and the map $Y'(\alpha)_{[A_0,h_0]}$ in* (6.18) *can be chosen to have the following properties*:

(1) *For fixed $[A,h] \in X'$, $Y'(\alpha)_{[A_0,h_0]}([A,h],\cdot)$ maps into $\{y \in \mathbf{R}^{n[A_0,h_0]}\colon |y| < \epsilon_0\}$ and surjectively onto an open neighborhood of* 0 *which contains $\{y \in \mathbf{R}^{n[A_0,h_0]}\colon |y| < z_1\cdot\epsilon_0\}$.*

(2) *The differential of $Y'(\alpha)_{[A_0,h_0]}([A,h],\cdot)\colon (z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\alpha)[A_0,h_0]} \to \mathbf{R}^{n[A_0,h_0]}$ has adjoint*

$$dY'(\alpha)_{[A_0,h_0]}([A,h],\cdot)^*\colon \mathbf{R}^{n[\sigma]} \to T(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\alpha)[A_0,h_0]}$$

which obeys

$$\inf_{0\neq y}(|dY'(\alpha)_{[A_0,h_0]}([A,h],\cdot)^*\cdot y|/|y|) > e_0.$$

(3) *On the complement of an open set in X' in which Z' has compact closure, and which has, itself, compact closure in X', the map $Y'(\alpha)_{[A_0,h_0]}$ depends only on the coordinates $(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\alpha)[A_0,h_0]}$.*

(4) *There exists a smooth map $\varsigma_0'(\alpha)\colon X' \to (3\cdot z_0^{-1}\cdot\epsilon_0, 1/3\cdot z_0\cdot\epsilon_0)^{k(\alpha)[A_0,h_0]}$ with the property that $Y'(\alpha)_{[A_0,h_0]}([A,h],\varsigma_0(\alpha)([A,h])) = 0$ for all $[A,h] \in X'$.*

Proof of Lemma 6.13. This is a straightforward perturbation argument which uses Lemma 6.12 and the contractibility of the set X'.

Since W is a compact subset of $\mathfrak{B}'(k,\eta)$, there exists a locally finite cover of U_0 by open sets $\{X_\sigma\colon \sigma \geq 1\}$, with each $X_\sigma \equiv X[A_\sigma,h_\sigma]$ as defined in the preceding paragraph for some $[A_\sigma,h_\sigma] \in U_0$. (Exhaust $\mathfrak{M}'(k,\eta)$ by a sequence of nested, compact sets, and use the analysis in [9] or in §5 of [22] to see that a locally finite cover can be constructed for $\mathfrak{M}'(k,\eta)$.) Let $Z_\sigma \subset X_\sigma$ and $Z'_\sigma \subset X'_\sigma \subset \mathfrak{B}'(k,\eta)$ be the corresponding open sets which are defined by

$[A_\sigma, h_\sigma]$. For each index σ, let

$$\{V(\sigma,\alpha), \Lambda(\sigma,\alpha), k(\sigma,\alpha), Y(\sigma,\alpha), \varsigma_0(\sigma,\alpha), Y'(\sigma,\alpha), \varsigma_0'(\sigma,\alpha) \colon \alpha \le q_0 \cdot J_1 \cdot J_1\}$$

and μ_σ, $n(\sigma)$, $\{\omega_{\sigma,n}\}_{n\le n[\sigma]}$ be as defined above using the point $[A_\sigma, h_\sigma] \in U_0$.

Due to Lemma 6.13, each pair $(Y'(\sigma,\alpha), \varsigma_0'(\sigma,\alpha))$ can be extended as smooth maps

$$\begin{aligned} Y'(\sigma,\alpha) &\colon \left(\bigcup_\sigma X'_\sigma\right) \times (z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)} \to \mathbf{R}^{n[\sigma]}, \\ \varsigma_0'(\sigma,\alpha) &\colon \bigcup_\sigma X'_\sigma \to (z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)} \end{aligned} \tag{6.19}$$

which agree with $(Y(\sigma,\alpha), \varsigma_0(\sigma,\alpha))$ on Z'_σ; and which obey

Lemma 6.14. *Let ϵ_0, z_0, z_1 and e_0 be as in Lemma 6.11. The maps $(Y'(\sigma,\alpha), \varsigma_0'(\sigma,\alpha))$ in (6.19) have the following properties:*

(1) *For fixed $[A,h] \in \bigcup_\sigma X'_\sigma$, $Y'(\sigma,\alpha)([A,h],\cdot)$ maps into $\{y \in \mathbf{R}^{n[\sigma]} \colon |y| < \epsilon_0\}$ and surjectively onto an open neighborhood of 0 which contains $\{y \in \mathbf{R}^{n[\sigma]} \colon |y| < z_1\cdot\epsilon_0\}$.*

(2) *The differential of $Y'(\sigma,\alpha)([A,h],\cdot)\colon (z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)} \to \mathbf{R}^{n[\sigma]}$ has adjoint $dY'(\sigma,\alpha)([A,h],\cdot)^*\colon \mathbf{R}^{n[\sigma]} \to T(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)}$ which obeys*

$$\inf_{0\ne y}(|dY'(\sigma,\alpha)([A,h],\cdot)^*\cdot y|/|y|) > e_0.$$

(3) *On the complement of X'_σ in $\bigcup_\sigma X'_\sigma$, the map $Y'(\sigma,\alpha)$ depends only on the coordinates $(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)}$.*

(4) *$Y'(\sigma,\alpha)([A,h], \varsigma_0'(\sigma,\alpha)([A,h])) = 0$ for all $[A,h] \in X'$.*

Proof of Lemma 6.14. This is immediate from Lemma 6.13.

Let $\{\psi_\sigma \colon \mathfrak{B}'(k,\eta) \to [0,1]\}$ be a smooth set of functions with two special properties. First, require that $\sum_\sigma \psi_\sigma|_{U_0} \equiv 1$; and second, require that for each σ, $\psi_\sigma \equiv 0$ on $\mathfrak{B}'(P)\backslash Z'_\sigma$.

Part 5: Redefining the parameters. Now, to specify the data $(\epsilon_0, r_1, r_2, R, \varepsilon, \{s(j,k,i)\})$ from Part 3, introduce the parameter space

$$\mathfrak{X} \equiv \left(\bigcup_\sigma X'_\sigma\right) \times_\sigma \left(\bigtimes_{\alpha\le q_0\cdot J_1\cdot J_2} (z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)}\right). \tag{6.20}$$

Note that the infinite product $\times_\sigma(\times_{\alpha\le q_0\cdot J_1\cdot J_2}(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)})$ has a natural manifold structure which is defined from the product structure. This is a Banach manifold structure; the tangent space to a point $\times_\sigma(\times_{\alpha\le q_0\cdot J_1\cdot J_2}\mathbf{R}^{k(\sigma,\alpha)})$, is isomorphic to l^∞.

Let $\varsigma \equiv (\varsigma(\sigma,\alpha))$ define a point in $\times_\sigma(\times_{\alpha\le q_0\cdot J_1\cdot J_2}(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)})$, where $\varsigma(\sigma,\alpha) \equiv (\varsigma_\sigma[j,k,i]\colon (j,k)\in\Lambda(\sigma,\alpha)$ and $i\le m))$ defines a point in $(\times_{\alpha\le q_0\cdot J_1\cdot J_2}(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)})$.

Fix a smooth function $\delta\colon \mathfrak{B}'(k,\eta)\to(0,\epsilon]$ and then define a function from $\mathfrak{X}$ to $(0,1]$ by sending $([A_0,h_0],\varsigma)$ to

$$(6.21)\quad s[j,k,i]([A_0,h_0],\varsigma)\equiv\sum_\sigma \psi_\sigma([A_0,h_0])\cdot\chi_\sigma(j,k)\cdot\varsigma_\sigma[j,k,i]+\delta([A_0,h_0])^2.$$

Here $\chi_\sigma(j,k)\equiv 1$ if $(j,k)\in\bigcup_\alpha\Lambda(\sigma,\alpha)$, and $\chi_\sigma(j,k)\equiv 0$ otherwise. (Note $\varsigma_\sigma[j,k,i]$ is only defined when $(j,k)\in\bigcup_\alpha\Lambda(\sigma,\alpha)$. For

$$(j,k)\notin\bigcup_\sigma\bigcup_\alpha\Lambda(\sigma,\alpha)\{(j,k)\}',$$

one has $s[j,k,i]([A_0,h_0],\varsigma)\equiv\delta([A_0,h_0])^2$.)

Over any compact subset of $(\bigcup_\sigma X'_\sigma)$, $s[j,k,i](\cdot)$ depends on only finitely many coordinates in $\times_\sigma(\times_{\alpha\le q_0\cdot J_1\cdot J_2}(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)})$. Thus, (6.21) defines a smooth function on $\mathfrak{X}$.

Specifying the set $\{s[j,k,i]\}$ in this way constructs from the map T of (6.15) a well-defined and a smooth map,

$$T'\colon\mathfrak{X}\to\mathfrak{B}'(k+c(G)\cdot J,\eta).$$

Part 6: Choosing the parameters. Use the observations of Part 5 to define $\{\mathfrak{f}_n\colon n\le N(7\cdot\mu_1)\}$ of (6.9) using the orbit $[A',h']\equiv T'([A_0,h_0],\varsigma)$ and an orthonormal basis $\{\omega_n\colon n\le N(8\cdot\mu_1)\}$ for $\operatorname{Range}\Pi(8\cdot\mu_1;A_0)$. Due to Lemma 6.4,

$$(6.22)$$
$$\mathfrak{f}_n=\sum_{\{(j,k,i)\}}\lambda^2[j,k,i]([A_0,h_0])(\omega_n([A_0,h_0])(x[j,k,i]),$$
$$X(f,h(x[j,k,i])([A_0,h_0]),h_1[j,k,i])\cdot P_+F_{A_1}(s))+\mathfrak{r}_n,$$

where $\mathfrak{r}_n$ is the tautologically defined remainder; it is estimated for $[A_0,h_0]\in W_\sigma$ by Lemma 6.4.

Using the partition of unity and (6.21), note that for $n\le N(7\cdot\mu_1)$, one has

$$(6.23)$$
$$\mathfrak{f}_n=\sum_{\{(j,k,i)\}}\varepsilon\cdot R\cdot J^{-1}\cdot\sum_\sigma\sum_{n_1}\langle\omega_n,\omega_{\sigma,n_1}\rangle_{L^2}\cdot(\psi_\sigma\cdot\chi_\sigma(j,k)\cdot\varsigma_\sigma[j,k,i]+\delta^2)$$
$$\cdot(\omega_{\sigma,n_1}([A_0,h_0])(x[j,k,i]),$$
$$X(\varphi(x[j,k,i]),f_0,[A_0,h_0],h_1[j,k,i])\cdot P_+F_{A_1}(s))+\sum_\sigma\psi_\sigma\cdot\gamma_n,$$

The preceding can be rewritten by decomposing the sum over indices $\{(j,k,i)]$ as

(6.24)

$$\begin{aligned}\mathfrak{f}_n = \varepsilon \cdot R \cdot J^{-1} \cdot \sum_\sigma \sum_\alpha \sum_{n_1} \langle \omega_n, \omega_{\sigma,n_1}\rangle_{L^2} \cdot \psi_\sigma \\ \cdot \Bigg(\sum_{(j,k)\in\Lambda(\sigma,\alpha)} \sum_i \varsigma_\sigma[j,k,i] \\ \cdot (\omega_{\sigma,n_1}([A_0,h_0])(x[j,k,i]), \\ X(\varphi(x[j,k,i]), f_0, [A_0,h_0], h_1[j,k,i]) \cdot P_+F_{A_1}(s)) \\ + (J_1 \cdot J_2 \cdot q_0)^{-1} \cdot \delta^2 \\ \cdot \sum_{\{(j,k,i)\}} (\omega_{\sigma,n_1}([A_0,h_0])(x[j,k,i]), \\ X(\varphi(x[j,k,i]), f_0, [A_0,h_0], h_1[j,k,i]) \cdot P_+F_{A_1}(s)) \\ + m \cdot (\varepsilon \cdot R \cdot q_0)^{-1} \cdot \sum_{n_2} \langle \omega_{\sigma,n_1}, \omega_{n_2}\rangle_{L^2} \cdot \mathfrak{r}_{n_2} \Bigg).\end{aligned}$$

(6.23) yields (6.24) because the sum over α contains precisely $J_1 \cdot J_2 \cdot q_0$ terms.

(6.24) makes evident the assertion that all $\mathfrak{f}_n$ vanish for $n \le N(4\cdot\mu_1)$ if, for every index σ and every index α, and for all $n \le N(\mu_\sigma)$

(6.25)

$$\begin{aligned}0 = \psi_\sigma \cdot \Bigg(\sum_{(j,k)\in\Lambda(\sigma,\alpha)} \sum_i \varsigma_\sigma[j,k,i] \\ \cdot (\omega_{\sigma,n}([A_0,h_0])(x[j,k,i]), \\ X(\varphi(x[j,k,i]), f_0, [A_0,h_0], h_1[j,k,i]) \cdot P_+F_{A_1}(s)) \\ + (J_1 \cdot J_2 \cdot q_0)^{-1} \cdot \delta^2 \\ \cdot \sum_{\{(j,k,i)\}} (\omega_{\sigma,n}([A_0,h_0])(x[j,k,i]), \\ X(\varphi(x[j,k,i]), f_0, [A_0,h_0], h_1[j,k,i]) \cdot P_+F_{A_1}(s)) \\ + m \cdot (\varepsilon \cdot R \cdot q_0)^{-1} \cdot \mathfrak{r}_{\sigma,n} \Bigg),\end{aligned}$$

where $\mathfrak{r}_{\sigma,n} \equiv \sum_{n'\le N(8\cdot\mu_1)} \langle \omega_{\sigma,n}, \omega_{n'}\rangle_{L^2} \cdot \mathfrak{r}_{n'}$. The utility of this last equation lies in the fact that when $[A_0,h_0] \in X'_\sigma$, then

$$\begin{aligned}
&\sum_{(j,k)\in\Lambda(\sigma,\alpha)} \sum_i \varsigma_\sigma[j,k,i] \\
(6.26)\qquad &\cdot (\omega_{\sigma,n}([A_0,h_0])(x[j,k,i]), \\
&\qquad X(\varphi(x[j,k,i]), f_0, [A_0,h_0], h_1[j,k,i]) \cdot P_+F_{A_1}(s)) \\
&= (\omega_{\sigma,n}([A_0,h_0]), Y(\sigma,\alpha)(\varsigma(\sigma,\alpha)) + \mathfrak{r}^1_{\sigma,n}([A_0,h_0],\varsigma(\sigma,\alpha)),
\end{aligned}$$

where the remainder $\mathfrak{r}^1_{\sigma,n}$ is a smooth function on

$$X'_\sigma \times \left(\bigtimes_{\alpha \le q_0\cdot J_1\cdot J_2} (z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)} \right)$$

which obeys, for $[A_0,h_0] \in X_\sigma$, the a priori bound

$$(6.27)\qquad |\mathfrak{r}^1_{\sigma,n}| \le z_0 \cdot r_2/r_1.$$

Here z_0 is a constant which is provided by the Riemannian metric via assertion (4) of Lemma 6.2.

For each index σ, let $\varphi_\sigma \colon \mathfrak{B}'(k,\eta) \to [0,1]$ be a smooth function with two special properties. First, require that $\sum_\sigma \varphi_\sigma|_{U_0} \equiv 1$; and second, require that for each σ, $\varphi_\sigma \equiv 0$ on $\mathfrak{B}'(k,\eta)\backslash Z'_\sigma$.

Introduce the infinite dimensional Banach space $\bigtimes_\sigma(\bigtimes_{\alpha\le q_0\cdot J_1\cdot J_2} \mathbf{R}^{n[\sigma]})$; the Banach space structure being defined by the obvious isomorphism with l^∞. Define a map

$$(6.28)\qquad \mathfrak{Y}\colon \mathfrak{X} \to \bigtimes_\sigma \left(\bigtimes_{\alpha\le q_0\cdot J_1\cdot J_2} \mathbf{R}^{n[\sigma]} \right)$$

by sending $([A_0,h_0],\varsigma)$ to $\mathfrak{Y} \equiv (\mathfrak{Y}(\sigma,\alpha)([A_0,h_0],\varsigma))$, where

$$(6.29)\qquad \mathfrak{Y}(\sigma,\alpha)([A_0,h_0]),\varsigma) \equiv Y'(\sigma,\alpha)([A_0,h_0],\varsigma) + \mathfrak{r}(\sigma,\alpha),$$

where $Y'(\sigma,\alpha)([A_0,h_0],\varsigma)$ is defined in (6.18) and where

(6.30)

$$\begin{aligned}
\mathfrak{r}(\sigma,\alpha) \equiv \varphi_\sigma \cdot \Sigma_n(\mathfrak{r}^1_{\sigma,n} + m\cdot(\varepsilon\cdot R\cdot q_0)^{-1}\cdot \mathfrak{r}_{\sigma,n} + (J_1\cdot J_2\cdot q_0)^{-1}\cdot\delta^2 \\
\cdot \sum_{\{(j,k,i)\}} (\omega_{\sigma,n}([A_0,h_0])(x[j,k,i]), X(\varphi(x[j,k,i]), \\
f_0, [A_0,h_0], h_1[j,k,i]) \cdot P_+F_{A_1}(s)).
\end{aligned}$$

Over any fixed compact set in $\bigtimes_\sigma X'_\sigma$, $\mathfrak{Y}(\sigma,\alpha)(\cdot)$ depends on only finitely many coordinates in $\bigtimes_\sigma(\bigtimes_{\alpha\le q_0\cdot J_1\cdot J_2}(z_0^{-1}\cdot\epsilon, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)})$; this means that $\mathfrak{Y}$ defines a smooth map.

Observe that when $[A_0,h_0] \in U_0$, the assertion that $\mathfrak{Y}([A_0,h_0]),\varsigma) = 0$ for some $\varsigma \in \bigtimes_\sigma(\bigtimes_{\alpha\le q_0\cdot J_1\cdot J_2}(z_0^{-1}\cdot\epsilon_0, z_0\cdot\epsilon_0)^{k(\sigma,\alpha)})$ implies that the set $\{\mathfrak{f}_n \colon n \le N(4\cdot\mu_1)\}$ vanishes for the orbit $T'([A_0,h_0],\varsigma)$.

For appropriate choices of $\varsigma \equiv \varsigma[A_0, h_0]$, r_1, r_2, ϵ_0, and R, $\mathfrak{Y}([A_0, h_0], \varsigma)$ will vanish. Indeed, choose $r_2/r_1 \equiv \epsilon$, $R \equiv \epsilon^{-1/64}$ and $J_1 = \epsilon^{-1/2}$. Now observe that the remainder in (6.30) obeys

$$|\mathfrak{r}| \leq z_0 \cdot \epsilon^{1/64}. \tag{6.31}$$

This estimate is a consequence of Lemmas 6.3–6.4 and (6.27).

To complete the proof of Proposition 6.1, note that (6.29) and (6.31) plus Lemma 6.14 plus the inverse function theorem provide the following data: First, they provide a constant $\epsilon_0 > 0$ which depends only on G and on the Riemannian metric. Second, when $\epsilon < \epsilon_0$, they provide an open neighborhood, $\mathfrak{N}$, of U_0 in $\mathfrak{B}'(k, \eta)$ with a smooth map,

$$\varsigma : \mathfrak{N} \to \bigtimes_{\sigma} \left(\bigtimes_{\alpha \leq q_0 \cdot J_1 \cdot J_2} (z_0^{-1} \cdot \epsilon_0, z_0 \cdot \epsilon_0)^{k(\sigma, \alpha)} \right) \tag{6.32}$$

which obeys

$$\mathfrak{Y}(\cdot, \varsigma(\cdot))|_{\mathfrak{N}} \equiv 0.$$

This last equation implies that $\{\mathfrak{f}_n : n \leq N(4 \cdot \mu_1)\}$ vanishes for each orbit $T'([A_0, h_0], \varsigma([A_0, h_0]))$ when $[A_0, h_0] \in U_0$. Thus,

$$\mathfrak{s}_{\mu_1}(T'([A_0, h_0], \varsigma([A_0, h_0]))) = 0$$

when $[A_0, h_0] \in U_0$.

Using (6.32), (6.21) defines the functions $\{s[j, k, i]\}$ over $\mathfrak{N}$. It is no task to extend the definition of the functions $\{s[j, k, i]\}$ from this domain to smooth functions which are defined on all of $\mathfrak{B}'(k, \eta)$ in such a way that the map T of (6.15) is well defined and agrees with $T'([A_0, h_0], \varsigma([A_0, h_0]))$ for $[A_0, h_0]$ in an open neighborhood of U_0. A specific extension will be considered in the next section. Also, a specific choice of the functions ε and δ will be made. With this extension made, the conclusions of Proposition 6.1 are established for some fixed $J \equiv J(U_0)$.

To obtain $T_{J'}$ for $J' > J$, one need only fix $J' - J$ distinct points on M which are disjoint from all balls $B[j]$ of Lemma 6.6, and disjoint from the base point x_0. Then, at each such point, glue the standard self-dual connection with scale size much less than ϵ^2/J'. A repetition of the preceding argument using Lemma 6.14 and the implicit function theorem provides a homotopy $T_{J'}$ which satisfies the requirements of Proposition 6.1 (see Proposition 6.2 in [21]).

7. Homotopy equivalences rel $\mathfrak{M}'(k, \eta)$

Fix a compact, oriented 4-dimensional manifold. In [13], Uhlenbeck proved that for all compact, simple Lie groups G, all moduli spaces $\{\mathfrak{M}'(k, \eta) \subset \mathfrak{B}'(k, \eta)\}$ are smoothly embedded submanifolds when the metric on TM is suitably generic. The generic set is a Baire set of C^p metrics for $p \geq 2$. (A Baire set is a countable intersection of open, dense sets, hence dense by Baire's theorem.) If the space of C^∞ metrics is given the usual topology as $\bigcap_p C^p$, then it is, in fact, a complete metric space. With this topology, Baire's theorem implies that there is a Baire set of C^∞ metrics on TM with the property that for all compact, simple Lie groups G, all moduli spaces $\{\mathfrak{M}'(k,\eta) \subset \mathfrak{B}'(k,\eta)\}$ are smoothly embedded submanifolds.

This Baire set of smooth metrics constitutes the set of metrics which are *G-good* for all G, where the term G-good is defined by

Definition 7.1. Let G be a compact, simple Lie group. A metric on TM is *G-good* when the operator $(P_- d_A)d_A^*$ is invertible on $L^2(P_-\Omega^2(\mathrm{Ad}\,P))$ whenever (P, A) is a pair of principal G-bundle $P \to M$ and self-dual connection A on P.

To discuss the relative topology of $(\mathfrak{B}'(k,\eta), \mathfrak{M}'(k,\eta))$ when (k,η) are characteristic classes for a principal G-bundle over M, it is convenient to distinguish metrics on TM as being G-good, or not. For G-good metrics, the following proposition describes the situation. Together, Proposition 7.2 implies Theorems 1 and 2 for a G-good metric.

Proposition 7.2. *Let G be a compact, simple Lie group, and let M be a compact, oriented 4-manifold with a G-good metric. Let (k,η) be characteristic classes for a principal G-bundle over M for which $\mathfrak{M}'(k,\eta)$ is nonempty. Then, for any $j \geq 0$, $\mathfrak{M}'(k + c(G)\cdot j, \eta)$ is nonempty, and there exists a map of pairs*

$$T(j)\colon (\mathfrak{B}'(k,\eta), \mathfrak{M}'(k,\eta)) \to (\mathfrak{B}'(k + c(G)\cdot j, \eta), \mathfrak{M}'(k + c(G)\cdot j, \eta))$$

with the following properties:

(1) *$T(j)\colon \mathfrak{B}'(k,\eta) \to \mathfrak{B}'(k + c(G)\cdot j, \eta)$ is a homotopy equivalence.*

(2) *If $j_1 \geq 0$ and $j_2 \geq 0$, then $T(j_1 + j_2)$ is homotopic to $T(j_2) \circ T(j_1)$ as maps of pairs*

$$(\mathfrak{B}'(k,\eta), \mathfrak{M}'(k,\eta)) \to (\mathfrak{B}'(k + c(G)\cdot(j_1+j_2), \eta), \mathfrak{M}'(k + c(G)\cdot(j_1+j_2), \eta)).$$

(3) *Let $z \in \pi_*(\mathfrak{B}'(k,\eta), \mathfrak{M}'(k,\eta))$ or $z \in H_*(\mathfrak{B}'(k,\eta), \mathfrak{M}'(k,\eta))$. There exists $J(z)$ such that for all $j \geq J(z)$, $T(j)_*(z) = 0$ in*

$$\pi_*(\mathfrak{B}'(k + c(G)\cdot j, \eta), \mathfrak{M}'(k + c(G)\cdot j, \eta))$$

or

$$H_*(\mathfrak{B}'(k + c(G)\cdot j, \eta), \mathfrak{M}'(k + c(G)\cdot j, \eta)).$$

The homotopy equivalences in the preceding proposition are modifications of the ones in §4. The proof of the proposition, which will be given shortly, provides the details.

When the metric on TM is not G-good, then there are moduli spaces $\mathfrak{M}'(k,\eta)$ which are not manifolds, or are not embedded in $\mathfrak{B}'(k,\eta)$. For these cases, the assertions of Proposition 7.2 must be modified. The modifications are made in Proposition 7.3 below, which establishes Theorems 1 and 2 in the general case.

Proposition 7.3. *Let G be a compact, simple Lie group, and let M be a compact, oriented Riemannian 4-manifold. Let (k,η) be characteristic classes for a principal G-bundle over M for which $\mathfrak{M}'(k,\eta)$ is nonempty. There exists $j(k)\geq 0$ such that for any $j\geq j(k)$, $\mathfrak{M}'(k+c(G)\cdot j,\eta)$ is nonempty, and there exists a map of pairs*

$$T(j,k)\colon (\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))\to(\mathfrak{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta))$$

with the following properties:

(1) *$T(j,k)\colon \mathfrak{B}'(k,\eta)\to\mathfrak{B}'(k+c(G)\cdot j,\eta)$ is a homotopy equivalence.*

(2) *If $j_1\geq j(k)$ and $j_2\geq j(k+c(G)\cdot j_1)$, then $T(k,k_2)$ is homotopic to $T(j_2,k+c(G)\cdot j_1)\circ T(j_1,k)$ as maps of pairs*

$$(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))\to(\mathfrak{B}'(k+c(G)\cdot(j_1+j_2),\eta),\mathfrak{M}'(k+c(G)\cdot(j_1+j_2),\eta)).$$

(3) *Let $z\in\pi_*(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))$ or $z\in H_*(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))$. There exists $J(z)\geq j(k)$ such that for all $j\geq J(z)$, $T(k,j)_*(z)=0$ in*

$$\pi_*(\mathfrak{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta))$$

or

$$H_*(\mathfrak{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta)).$$

The remainder of this section contains the proofs for Propositions 7.2 and 7.3.

Proof of Proposition 7.2. In §4 (see Proposition 4.1 and (4.10)) a homotopy equivalence $T\colon\mathfrak{B}'(k,\eta)\to\mathfrak{B}'(k+c(G),\eta)$ was constructed. Let $\mathfrak{T}'\equiv\{T'\colon\mathfrak{B}'(k,\eta)\to\mathfrak{B}'(k+c(G),\eta)\colon T'$ is homotopic to $T\}$. This is a connected family of homotopy equivalences from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G),\eta)$. Given $\epsilon>0$ and a smooth function $\mathfrak{f}\colon\mathfrak{M}'(k,\eta)\to(0,\epsilon)$, one can require of $T'\in\mathfrak{T}'$ that $\mathfrak{a}(T'(b))<\mathfrak{f}(b)$ for all $b\in\mathfrak{M}'(k,\eta)$. (Make a suitable choice of the functions $s(\cdot)$, $t(\cdot)$ in (4.1).)

By appealing to (6.6) and Lemma 5.4, one observes that for suitable $\mathfrak{f}$ and $\epsilon>0$, there will exist a homotopy $\Psi\colon[0,1]\times\mathfrak{B}'(k+c(G),\eta)\to\mathfrak{B}'(k+c(G),\eta)$ which fixes $\mathfrak{M}'(k+c(G),\eta)$ and $\{b\in\mathfrak{B}'(k+c(G),\eta)\backslash\mathfrak{B}'_\epsilon(k+c(G),\eta)$ and which retracts $\{b\in\mathfrak{B}'(k+c(G),\eta)\colon\mathfrak{a}(b)<\mathfrak{f}(b)\}$ into $\mathfrak{M}'(k+c(G),\eta)$.

Define $\mathfrak{T}$ to be the set of homotopy equivalences from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G),\eta)$ of the form $\Psi(1,T'(\cdot))$ for $T'\in\mathfrak{T}'$. Every homotopy equivalence in $\mathfrak{T}$ is a map of pairs

$$(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))\to(\mathfrak{B}'(k+c(G),\eta),\mathfrak{M}'(k+c(G),\eta)).$$

Composition defines for each $j > 0$ a set $\mathfrak{T}(j)$ of homotopy equivalences from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G)\cdot j,\eta)$ which map pairs $(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))$ to pairs $(\mathfrak{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta))$.

The fact that $\mathfrak{T}(j)$ was defined to be the composition of homotopies in $\mathfrak{T}$ means that assertion (2) of the proposition follows automatically. By restricting somewhat the choice of the functions $(s(\cdot),t(\cdot))$ in (4.1), an appeal to (6.6) and Lemma 5.4 shows that the family $\mathfrak{T}$ is connected as maps of pairs; hence, so is $\mathfrak{T}(j)$ for all $j > 0$.

Given $z \in \pi_*(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))$ or $z \in H_*(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))$, and given $\epsilon > 0$, §4 (see Proposition 4.2) finds $J(z,\epsilon)$ such that $T(j)_* z$ factors through the inclusion $(\mathfrak{B}'_\epsilon(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta)) \to (\mathfrak{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta))$ whenever $j > J(z,\epsilon)$ and $T(j) \in \mathfrak{T}(j)$.

Let $\mathfrak{s}_{\mu_0(\cdot)}$ denote the obstruction section of (5.2). By choosing $\epsilon > 0$ appropriately, and by choosing the functions ε in (6.4) and δ in (6.21) appropriately, §6 (see Proposition 6.1) finds $J(z)$ such that $T(j)_* z$ factors through the inclusion $(\mathfrak{s}^{-1}_{\mu_0(\cdot)}(0),\mathfrak{M}'(k+c(G)\cdot j,\eta)) \to (\mathfrak{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta))$ whenever $j > J(z,\epsilon)$ and $T(j) \in \mathfrak{T}(j)$. The proof of assertion (3) of Proposition 7.2 follows from this last fact and Lemmas 5.3 and 5.4.

Proof of Proposition 7.3. §6 (see Proposition 6.1) provides $j(k)$ and it provides a family of maps $\mathfrak{T}'(j(k),k)$ which is composed of certain homotopy equivalences from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G)\cdot j(k),\eta)$ which map $\mathfrak{M}'(k,\eta)$ into $\mathfrak{s}^{-1}_{\mu_0(\cdot)}(0)$. In this construction, a fixed choice of gluing points is determined in Part 2 of §6. The family is defined by taking different choices for the functions ε in (6.14) and δ in (6.21). By appealing to Lemma 5.3, a family of maps $\mathfrak{T}(j(k),k)$ is obtained; each map in $\mathfrak{T}(j(k),k)$ is a homotopy equivalence between $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G)\cdot j(k),\eta)$ which maps the pair $(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))$ to $(\mathfrak{B}'(k+c(G)\cdot j(k),\eta),\mathfrak{M}'(k+c(G)\cdot j(k),\eta))$.

For $j > j(k)$, §6 and Lemma 5.3 provide a family of maps, $\mathfrak{T}(j,k)$, which is composed of maps of pairs from $(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))$ to $(\mathfrak{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta))$ that are homotopy equivalent as a map from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G)\cdot j,\eta)$. This family is constructed from $\mathfrak{T}'(j(k),k)$ by first adding additional gluing points to boost the Pontrjagin number (as per (4.10)), but with the gluing parameters (the choice of ς and δ in (6.21)) taken very much smaller than those which defined the maps in $\mathfrak{T}'(j(k),k)$. Then, via slight readjustments of the functions ς, and δ at the original gluing sites for the maps in $\mathfrak{T}'(j(k),k)$ (see Part 6 of §6), a family of homotopy equivalences $\mathfrak{T}'(j,k)$ from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G)\cdot j,\eta)$ is constructed which maps $\mathfrak{M}'(k,\eta)$ into $\mathfrak{s}^{-1}_{\mu_0(\cdot)}(0)$. Using Lemma 5.3, this family is deformed to a family of homotopy equivalences $\mathfrak{T}(j,k)$ from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k+c(G)\cdot j,\eta)$ which is composed of maps of pairs $(\mathfrak{B}'(k,\eta),\mathfrak{M}'(k,\eta))$ to $(\mathfrak{B}'(k+c(G)\cdot j,\eta),\mathfrak{M}'(k+c(G)\cdot j,\eta))$.

Consider assertion (2) of the proposition. Let $j \geq j(k)$ and let $j_1 \geq j(k + c(G) \cdot j)$. Let $T \in \mathfrak{T}(j,k)$, and let $T_1 \in \mathfrak{T}(j_1, k + c(G) \cdot j)$. One can assume that the functions ε of (6.14) and δ of (6.22) are as small as desired. By making these functions sufficiently small, one can show that $T_1 \circ T$ is homotopic to some $T' \in \mathfrak{T}(j + j_1, K)$ through homotopy equivalences of $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k + c(G) \cdot (j + j_1), \eta)$ which send $\mathfrak{M}'(k,\eta)$ into $\mathfrak{M}'(k + c(G) \cdot (j + j_1), \eta)$. Lemma 6.14 with (6.28)–(6.30) and the implicit function theorem provide the crucial tools; the argument is straightforward and omitted.

Consider now what happens upon the specification of a class $z \in \pi_*(\mathfrak{B}'(k,\eta), \mathfrak{M}'(k,\eta))$ or $z \in H_*(\mathfrak{B}'(k,\eta), \mathfrak{M}'(k,\eta))$. Let $W \subset \mathfrak{B}'(k,\eta)$ be a compact set which represents z. Given $\epsilon > 0$, the inclusion map defines z as a relative class for the pair $(\mathfrak{B}'(k,\eta), \mathfrak{B}'_\epsilon(k,\eta))$. §4 (see Proposition 4.2) provides $J(z,\epsilon)$ and, for $j \geq J(z,\epsilon)$, a homotopy equivalence $T'\colon \mathfrak{B}'(k,\eta) \to \mathfrak{B}'(k + c(G) \cdot j, \eta)$ with the property that T maps the pair $(\mathfrak{B}'(k,\eta), \mathfrak{B}'_\epsilon(k,\eta))$ into $(\mathfrak{B}'(k + c(G) \cdot j, \eta), \mathfrak{B}'_{z_0 \cdot \epsilon}(k + c(G) \cdot j, \eta))$ and has z in the kernel of the induced map T_* on the relative homotopy or homology groups. It is easy to homotope T' of §4 to some $T \in \mathfrak{T}(j,k)$ through homotopy equivalences from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k + c(G) \cdot j, \eta)$ which map the pair $(\mathfrak{B}'(k,\eta), \mathfrak{B}'_\epsilon(k,\eta))$ into $(\mathfrak{B}'(k + c(G) \cdot j, \eta), \mathfrak{B}'_{z_0 \cdot \epsilon}(k + c(G) \cdot j, \eta))$. Indeed, choose any $T \in \mathfrak{T}(j, K)$. The homotopies T and T' are constructed in similar ways. The number of gluing sites are the same (and equal to j). However, the positions of the gluing sites might be different. Pair them up, one by one. For each pair, choose a path between the two points; require that the set of paths are disjoint. By decreasing the value of the functions ε in (6.14) and δ in (6.21) which define T, one can deform T into T' by moving the gluing sites for T along the chosen path to the respective paired site for T'.

Since T defines a homotopy equivalence from $\mathfrak{B}'(k,\eta)$ to $\mathfrak{B}'(k + c(G) \cdot j, \eta)$ which maps $\mathfrak{M}'(k,\eta)$ into $\mathfrak{M}'(k + c(G) \cdot j, \eta)$, the construction of the preceding paragraph shows that $T_* z$ is a well-defined relative class for the pair $(\mathfrak{B}'(k + c(G) \cdot j, \eta), \mathfrak{M}'(k + c(G) \cdot j, \eta))$ which factors through the inclusion of $\mathfrak{B}'_{z_0 \cdot \epsilon}(k + c(G) \cdot j, \eta)$ into $\mathfrak{B}'(k + c(G) \cdot j, \eta)$.

The constructions in §6 plus Lemma 6.3 provide $\epsilon > 0$; and for $j > J(z,\epsilon)$, they provide $J(z,j)$; and for all $j_1 > J(z,j)$ they provide a homotopy equivalence $T_1\colon \mathfrak{B}'(k + c(G) \cdot j, \eta) \to \mathfrak{B}'(k + c(G) \cdot (j + j_1), \eta)$ with the following properties: T_1 maps $\mathfrak{M}'(k + c(G) \cdot j, \eta)$ into $\mathfrak{M}'(k + c(G) \cdot (j + j_1), \eta)$; and $T_{1*} T_* z$ is annihilated in the relative homotopy or homology of the pair $(\mathfrak{B}'(k + c(G) \cdot (j + j_1), \eta), \mathfrak{M}'(k + c(G) \cdot (j + j_1), \eta))$.

With an appropriate choice for the functions ε in (6.14) and δ in (6.21), it is a straightforward process to homotope $T_1 \circ T$ to a homotopy in $\mathfrak{T}(j + j_1, k)$; this step is an application of Lemma 6.14, (6.28)–(6.30) and the implicit function theorem.

This completes the proof of assertions (1)–(3) of Proposition 7.3.

Appendix. Eigenvalue estimates

Let $P \to M$ be a principal G-bundle, and let $[A] \in \mathfrak{B}(P)$. Define an L^2-eigenvector of $P_-d_A(P_-d_A)^*$ with eigenvalue λ to be an L^2-section ω of $P_-\Omega^2(\operatorname{Ad} P)$ which solves the following differential equation:

$$P_-d_A(P_-d_A)^*\omega = \lambda \cdot \omega. \tag{A.1}$$

Fix $E \geq 0$. The purpose of this section is to provide an a priori estimate for the number of linearly independent L^2-eigenvectors of the operator $P_-d_A(P_-d_A)^*$ with eigenvalue less than or equal to E. The estimate comes from Proposition A.1, below, and it depends on the orbit $[A]$ and the precise principal G-bundle only through the L^2-norm of P_-F_A.

Proposition A.1 asserts a more general result. To state Proposition A.1, consider a finite dimensional Hilbert space V on which G acts isometrically via $\rho\colon G \to \operatorname{End} V$. Let $K \to M$ be a vector bundle which is associated to the principal SO(4) bundle of orthonormal frames in TM. If A is a connection on P, then A defines a connection on the vector bundle $P \times_\rho V$, and together with the Levi-Civita connection on the frame bundle, a connection is defined on the vector bundle $((P \times_\rho V) \otimes K)$. Denote by $\nabla_A\colon L^2_1((P \times_\rho V) \otimes K) \to L^2((P \times_\rho V) \otimes K \otimes T^*M)$ the associated covariant derivative. Let T be an L^2_1-section of the vector bundle of self-adjoint endomorphisms of $(P \times_\rho V) \otimes K$. Then the assignment of $\omega \in L^2_1((P \times_\rho V) \times K)$ to

$$\mathfrak{t}_A\omega \equiv \nabla_A * \nabla_A\omega + T \cdot \omega \tag{A.2}$$

defines an essentially selfadjoint, unbounded operator on $L^2((P \times_\rho V) \otimes K)$ with dense domain $L^2_2((P \times_\rho V) \otimes K)$ (use (3.4)).

Standard elliptic theory can be used to show that $\mathfrak{t}_A$ has pure point spectrum with finite multiplicities and with no accumulation point. The spectrum lies on the real axis and is bounded from below.

Proposition A.1. *Let G be a compact Lie group, and let V be a finite dimensional Hilbert space on which G acts isometrically via $\rho\colon G \to \operatorname{End} V$. Let M be a compact, oriented, Riemannian 4-manifold and let $K \to M$ be a vector bundle which is associated to the principal* SO(4) *bundle of orthonormal frames in TM. Given $E \in \mathbf{R}$ and $\alpha \geq 0$, there exists a constant $c_0(E, \alpha)$ with the following properties: Let $P \to M$ be a principal G-bundle, and let $[A] \in \mathfrak{B}(P)$. Let $T \in L^2_1(\operatorname{End}((P \times_\rho V) \otimes K))$ with $\|T\|_{L^2} \leq \alpha$. The number* *of linearly independent eigenvectors of* $\mathfrak{t}_A$ *with eigenvalue less than* E *is no greater than* $c_0(E, \alpha)$.

A similar result has been announced by [4] using a theorem of Cwickel-Lieb-Rosenbljum [18] (see also [16]).

The remainder of this appendix is concerned with the proof of Proposition A.1.

The proof of Proposition A.1 begins with the following observation:

Lemma A.2. *Let* M *be a compact Riemannian manifold, and let* r_0, $\epsilon > 0$ *and* $\alpha < \infty$ *be given. There exists an integer* $N(r_0, \epsilon, \alpha)$ *with the following significance: Let* f *be an* L^2*-function on* M *with* $\|f\|_{L^2} \le \alpha$. *There exists some* $N < N(r_0, \epsilon, \alpha)$ *functions* $\{u[\sigma]\}_{\sigma \le N} \subset L^2(M)$ *with the properties that*

(1) $\sum_\sigma u[\sigma] \equiv f$.

(2) *If* $\sigma \ne \sigma'$ *then* $u[\sigma] \cdot u[\sigma'] = 0$ *almost everywhere on* M.

(3) *The support of* $u[\sigma]$ *is contained in a ball* $B(x[\sigma], r[\sigma])$ *with* $r[\sigma] < r_0$.

(4) *For* $\sigma_0 \le N$, $\int_{B(x[\sigma_0], 2\cdot r[\sigma_0])} \sum_{\sigma \ge \sigma_0} u[\sigma]^2 < \epsilon^2$.

Proof of Lemma A.2. First, observe that there exists $z < \infty$ which depends only on the dimension of M and which has the following significance: Suppose that $r_0 > 0$ is less than 1/16 times the injectivity radius of M. Let $x \in M$ and $r \in (0, r_0)$ be given. Then the ball $B(x, 2 \cdot r)$ is covered by no more than z balls $\{B(y, r) : y \in B(x, 2 \cdot r)\}$.

Given $r_0 > 0$, but less than 1/16 times the injectivity radius of M, there exists a cover of M by $N(r_0/2)$ open ball $\{B(x[j], r_0/2)\}$ of radius $r_0/2$.

Define $f_0 \equiv f$. Suppose that for $n > 0$, a function $f_n \in L^2(B(x[1], r_0)$ has been defined. For each $x \in B(x[1], r_0/2)$, let $r(f_n, x)$ denote the supremum over the set of $r \in (0, r_0/2]$ with the property that $\|f_n\|_{L^2; B(x,r)} \le \epsilon/(2 \cdot z)$. Let x_n minimize $r(f_n, \cdot)$ as a function on $B(x[1], r_0)$, and let $r_n \equiv r(f_n, x_n)$. Let χ_n denote the characteristic function for the ball $B(x_n, r_n)$, and let $f_{n+1} \equiv f_n - \chi_n \cdot f_n$. Note that this recursive definition of L^2-functions on $B(x_0, r_0)$ must terminate after at most $\|f\|_{L^2}/\epsilon$ steps. Note as well that

$$\int_{B(x_n, 2\cdot r_n)} f_n^2 < \epsilon. \tag{A.3}$$

Now, define $u[\sigma \equiv (1, n)] \equiv \chi_n f_n$. For given index j, if the functions $\{u[\sigma = (j, n)]\}$ have been constructed, construct the set $\{u[\sigma \equiv (j+1, n)]\}$ by repeating the procedure above with $f - \sum_{j' \le j \text{ and } n} u[\sigma]$ replacing f and with $B(x[j], r_0)$ replacing $B(x[1], r_0)$.

Suppose $E \in \mathbf{R}$ is given. Let $\omega \in L^2_2((P \times_\rho V) \otimes K)$ be a linear combination of eigenvectors of $\mathfrak{t}_A$ with eigenvalues less than E. Then

$$\langle \nabla_A \omega, \nabla_A \omega \rangle_{L^2} - \langle \omega, |T| \cdot \omega \rangle_{L^2} \le E \cdot \langle \omega, \omega \rangle_{L^2}. \tag{A.4}$$

Use Lemma A.2 to construct the set of L^2-functions $\{u[\sigma]\}$ using the L^2-function $f \equiv |T|$.

Let $\beta(\cdot)\colon [0,\infty) \to [0,1]$ be a smooth function which is identically one on $[0,1]$ and which vanishes on $[2,\infty)$. Define for each $(x,r) \in M \times (0,1]$ the function $\beta_{(x,r)}(\cdot) \equiv \beta(\mathrm{dist}(x,\cdot)/r)$. For each index σ, set $\beta[\sigma] \equiv \beta_{(x[\sigma],r[\sigma])}$ with $(x[\sigma], r[\sigma])$ provided by Lemma A.2. Set

$$v[\sigma] \equiv \left(\beta[\sigma] \cdot \prod_{\sigma'<\sigma}(1-\beta[\sigma'])\right)^{1/2}, \tag{A.5}$$

and note that $\sum_\sigma v[\sigma]^2 \equiv 1$, and if $\sigma' < \sigma$, then $u[\sigma'] \cdot v[\sigma] \equiv 0$.

Then, rewrite (A.4) by inserting the partition of unity $\{v[\sigma]^2\}$ as

$$\langle \nabla_A\omega, \nabla_A\omega\rangle_{L^2} - \sum_\sigma \langle v[\sigma]\cdot\omega, |T|\cdot v[\sigma]\cdot\omega\rangle_{L^2} \le E \cdot \langle\omega,\omega\rangle_{L^2}. \tag{A.6}$$

Due to Lemma A.2, this last equation can be written as

$$\langle \nabla_A\omega, \nabla_A\omega\rangle_{L^2} - \sum_\sigma \left\langle v[\sigma]\cdot\omega, \sum_{\sigma'\ge\sigma} u[\sigma]\cdot v[\sigma]\cdot\omega\right\rangle_{L^2} \le E\cdot\langle\omega,\omega\rangle_{L^2}, \tag{A.7}$$

and, therefore, due to assertion (4) of Lemma A.2 and Holder's inequality,

$$\langle \nabla_A\omega, \nabla_A\omega\rangle_{L^2} - \epsilon\cdot\sum_\sigma \|v[\sigma]\cdot\omega\|_{L^4}^2 \le E\cdot\langle\omega,\omega\rangle_{L^2}. \tag{A.8}$$

Since $v[\sigma]\cdot\omega$ has compact support in the ball $B(x[\sigma], 2\cdot r[\sigma])$, the $L_1^2 \to L^4$ Sobolev space inclusion with (3.4) imply that

$$\begin{aligned} \|v[\sigma]\cdot\omega\|_{L^4}^2 &\le z\cdot\|d|v[\sigma]\cdot\omega|\|_{L^2}^2 \\ &\le 2\cdot z\cdot\|v[\sigma]\cdot\nabla_A\omega\|_{L^2}^2 + \||dv[\sigma]|\cdot\omega|\|_{L^2}^2. \end{aligned} \tag{A.9}$$

Together, (A.8) and (A.9) imply that

$$(1-2z\epsilon)\cdot\langle\nabla_A\omega,\nabla_A\omega\rangle_{L^2} - 2z\epsilon\cdot\sum_\sigma \||dv[\sigma]|\cdot\omega|\|_{L^2}^2 \le E\cdot\langle\omega,\omega\rangle_{L^2}. \tag{A.10}$$

Here, since each $r[\sigma]$ is less than $1/16$ times the injectivity radius of M, the constant z is fixed, independent of M.

To evaluate (A.10), observe that $|d\beta[\sigma]| < z/r[\sigma]$ and it has support in $B(x[\sigma], 2\cdot r[\sigma])$. Thus, if ϵ is taken less than $z/4$, then (A.10) implies that

$$\begin{aligned} &\sum_\sigma (1/N(r_0,\epsilon,\alpha)\cdot\langle\nabla_A\omega[\sigma],\nabla_A\omega[\sigma]\rangle_{L^2;B(x[\sigma],4r[\sigma])} \\ &\qquad - z\cdot(E + N(r_0,\epsilon,\alpha)^2/r[\sigma]^2)\cdot\langle\omega[\sigma],\omega[\sigma]\rangle_{L^2;B(x[\sigma],4r[\sigma])}) \le 0, \end{aligned} \tag{A.11}$$

where $\omega[\sigma] \equiv \beta_{(x[\sigma],2\cdot r[\sigma])}\cdot\omega$.

To analyze (A.11), it is necessary to digress and discuss the trace Laplacian, $\nabla_A^*\nabla_A$. For this purpose, let $B \subset M$ be a ball of radius less than the 1/4 times injectivity radius of M, and let $L^2_{2;0}((P\times_\rho V)\otimes K|_B)$ denote the Hilbert space of L^2_2-sections of $(P \times_\rho V) \otimes K$ over B which vanish on ∂B.

The trace Laplacian $\nabla_A^*\nabla_A$ defines an unbounded, self-adjoint operator Δ_A on $L^2((P\times_\rho V)\otimes K|_B)$ with dense domain $L^2_{2;0}((P\times_\rho V)\otimes K|_B)$ which is called the Dirichlet Laplacian.

Lemma A.3. *Let M be a compact, Riemannian manifold. There exists $r_1 > 0$ and, given $\lambda \geq 0$, there exists $N(\lambda) < \infty$ which have the following significance: Let $B \subset M$ be a ball of radius $r < r_1$. Let (V, ρ, P, K) be as in Proposition* A.1. *Let A be a connection on P. Then there are at most $N(\lambda)$ eigenvectors of the Dirichlet Laplacian Δ_A on $L^2((P\times_\rho V)\otimes K|_B)$ with eigenvalue less than λ/r^2.*

This lemma will be proved shortly.

Proof of Proposition A.1 *assuming Lemma* A.3. Choose r_0 in (A.11) to be less than r_1 of Lemma A.3. Choose λ in Lemma A.3 to be equal to $32 \cdot z \cdot N(r_0, \epsilon, \alpha) \cdot (E \cdot r_0^2 + N(r_0, \epsilon, \alpha)^2)$, with the constant z as in (A.11). Suppose that there existed more than $N(\lambda)\cdot N(r_0, \epsilon, \alpha)$ eigenvectors of $\mathfrak{t}_A$ with eigenvalue less than E. Then, one could find a nontrivial linear combination of these eigenvectors, ω, with the following property: For each of the less than $N(r_0, \epsilon, \alpha)$ indices σ, the section $\omega[\sigma] \in (L^2_{2;0}((P\times_\rho V)\otimes K|_{B(x[\sigma],4r[\sigma])}))$ is L^2-orthogonal to the span of the eigenvectors of the Dirichlet Laplacian Δ_A on $L^2((P\times_\rho V)\otimes K|_{B(x[\sigma],4r[\sigma])})$ with eigenvalue less than λ. For such ω, (A.11) would yield an immediate contradiction.

Proof of Lemma A.3. The proof mimics an argument in [7]. To begin, construct the heat kernel for the Dirichlet Laplacian; denoted by $k_A(t;\cdot,\cdot)$. This kernel can be written explicitly using the eigenvectors to the Dirichlet Laplacian,

$$k_A(t;x,y) \equiv \sum e^{-t\cdot\lambda(v)/r^2} v(x)\otimes v(y), \tag{A.12}$$

where the sum is over an L^2-orthonormal set $\{v\}$ of Dirichlet eigenvectors of Δ_A. Here the number $\lambda(v)/r^2$ is the eigenvalue for the eigenvector v. Observe that when the point y is in the interior of B, the heat kernel obeys the heat equation,

$$(\partial/\partial t + \Delta_A)k_A(\cdot;\cdot,y)|_{t>0} = 0 \quad \text{with } k_A(0;\cdot,y) \equiv \delta(\cdot,y)\cdot \mathbf{I}. \tag{A.13}$$

Here $\delta(\cdot,y)$ is the Dirac delta function with support at y, and

$$\mathbf{I} \in \mathrm{Hom}\,((P\times_\rho V)\otimes K|_y, (P\times_\rho V)\otimes K|_y)$$

is the identity homomorphism.

Note that on ∂B, $k_A(t;\cdot,y)$ vanishes. Thus, by Kato's inequality ((3.4)) and the maximum principle,

$$|k_A(t;x,y)| \leq m \cdot |k_0(t;x,y)| \quad \text{for } t > 0 \text{ and for } x, y \in B. \tag{A.14}$$

Here $k_0(t;\cdot,\cdot)$ is the Dirichlet heat kernel for the scalar Laplacian d^*d on $L^2(B)$, and m is the fiber dimension of the vector bundle $(P \times_\rho V) \otimes K$.

Together, (A.12) and (A.14) imply that

$$\begin{aligned} &m \cdot \int k_0(t;x,x) \cdot d\,\mathrm{vol}(x) \\ &\quad \geq \int \mathrm{tr}(k_A(t;x,x)) \cdot d\,\mathrm{vol}(x) \geq N(A,\lambda) \cdot e^{-t\cdot\lambda/r^2}. \end{aligned} \tag{A.15}$$

Here, tr is the trace on $\mathrm{Hom}((P \times_\rho V) \otimes K|_x, (P \times_\rho V) \otimes K|_x)$, and $N(A,\lambda)$ is the number of linearly independent eigenvectors of the ∇_A with eigenvalue less than λ/r^2.

For the proof of Lemma A.3, it remains yet to bound the left-hand side of (A.15). Since M is compact, there exists $r_2 > 0$ so that a Gaussian coordinate system which is centered at a given point in M covers the ball of radius r_2 about that point. Gaussian coordinates identify this ball with a ball which is centered at the origin of Euclidean space.

There exists $r_3 \equiv (0, r_2)$ such that if B is a ball of radius $r < r_3$ in M, then $K_0(t;x,x)$ is uniformly estimated by the heat kernel for the Euclidean metric's Dirichlet Laplacian [5]. The result is that there exists $\delta_0 > 0$ which depends on the Riemannian metric and which is such that when $t < \delta_0$, the left-hand side of (A.15) is bounded by $z \cdot r^4/t^2$. Take $\lambda > 1$, and take $t = r^2/\lambda$, then (A.15) implies that

$$z \cdot m \cdot \lambda^2 \geq N(A,\lambda).$$

This estimate implies the lemma.

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